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Sphere-Contact Singularities of Cuspidal Edge–Cusp Pairs

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Abstract

In this study, we examine sphere contact for a cuspidal edge–cusp pair (X, B), where $X \subset \mathbb{R}^3$ is a cuspidal edge and $B \subset X$ is a distinguished curve whose germ at the singular point is a $(2,3)$-type cuspidal space curve. Let $f\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a parametrization of X, and let $\alpha\colon (\mathbb{R},0) \to (\mathbb{R}^2,0)$ be a source curve whose image under f is the distinguished cusp B. We consider the restricted distance squared family $D_{(X,B)}(s,q) = \|\beta(s) - q\|^2, q \in \mathbb{R}^3$, which is determined by the singular space curve β, where $\beta = f \circ \alpha$. We describe its contact strata with the aim of also reflecting the role of β as a distinguished curve on a cuspidal edge, together with its chosen source curve. We prove that centres with a contact of order at least 2 form a plane, while those with contact of order at least 3 form a line; these are also known as the singular normal plane and the singular binormal line of the pair, respectively. We then show that there exists a unique centre with a contact in the order of at least 4 if and only if the singular torsion does not vanish. In this case, the osculating centre is $q_{\mathrm{osc}}(X,B) = \beta(0) + \frac{3}{\kappa_{\mathrm{sing}}(\beta,0)\tau_{\mathrm{sing}}(\beta,0)}\mathbf{b}_{\mathrm{sing}}(\beta,0)$, where $\mathbf{b}_{\mathrm{sing}}(\beta,0)$ is the unit singular binormal vector of β. Also, we determine the exact A_2- and A_3-strata and obtain a criterion for fifth-order contact at the osculating centre, with the lower strata determined by the 3-jet of the distinguished cusp. The fourth-order condition takes the form of a mixed jet expression in adapted normal form, involving both the cuspidal edge and the chosen source curve.

Keywords: cuspidal edge; singular space curve; distance squared function; sphere contact; osculating sphere

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1. Introduction

Distance squared functions are a classical tool for studying contact with spheres. In a regular setting, they recover osculating objects, focal sets, and higher-order tangencies (see[1–4]). More generally, Montaldi's contact framework relates singularity type to the order of contact between submanifolds and model objects [5].

For singular surfaces in $\mathbb{R}^3$, the same viewpoint is equally natural. Cuspidal edges are among the standard generic singularities of fronts in $\mathbb{R}^3$ [6–8]. Their local differential geometry has been examined from multiple angles (see [6–10]). However, singular curves in space also have a rich differential geometry. This theory is classical and, at the very least, goes back to Sasai's work on analytic space curves with singularities [11]. Later studies by Shiba and Umehara clarified the normalized curvature picture for cusps in the planar case [12]. For $(2,3)$-type space cusps, cuspidal curvature and singular torsion

were introduced by Martins and Saji [13], which were also discussed by Umehara, Saji, and Yamada [14]. These invariants provide a standard framework for studying singular curves and related surface geometry [13,14]. Related osculating-sphere and focal geometry for singular space curves has been studied via evolutes and focal surfaces of framed immersions [15]. We contribute an explicit description of the sphere-centre strata obtained from the restricted distance squared family at the distinguished cusp. In particular, we characterize the existence and uniqueness of a fourth-order centre by the nonvanishing of singular torsion and derive a fifth-order contact criterion, which we express in adapted coordinates for the cuspidal edge and the chosen source curve.

These contact results apply to any $(2,3)$-type space cusp. The pair formulation describes their realization through the coefficients of the cuspidal edge and the chosen source curve. Once the parametrized image cusp β is fixed, the restricted family contains no additional information about the ambient surface.

We formulate this sphere-contact problem for a distinguished cusp on a cuspidal edge. Naturally, this leads to the study of a pair (X, B), where X is a cuspidal edge and $B \subset X$ is a distinguished singular curve. This viewpoint is advantageous because it allows us to interpret the sphere-contact strata as geometric objects naturally attached to the pair; it also shows how the invariants of the cusp can be expressed in terms of the ambient cuspidal edge and the chosen source curve.

Such pairs arise naturally in several settings. In particular, cuspidal edges with distinguished singular curves appear in bifurcation diagrams of certain caustics [16]. This viewpoint appears, for example, in Zakalyukin's contact flag singularities [17,18] and in related corner flag settings [19]. A closely related perspective comprises studies of submersions and height functions on a cuspidal edge together with a distinguished curve, rather than on the cuspidal edge alone [16,20]. This study elaborates on this viewpoint for sphere contact. More precisely, we consider a pair (X, B), where $X = f((\mathbb{R}^2, 0)) \subset \mathbb{R}^3$ is a cuspidal edge germ, and $B \subset X$ is a distinguished curve whose image germ is a $(2,3)$-type cuspidal space curve.

Let $\alpha\colon (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a source curve whose image under f is B. The associated restricted distance squared family is

$$D_{(X,B)}(s, q) = \|\beta(s) - q\|^2, \qquad q \in \mathbb{R}^3,$$

where $\beta = f \circ \alpha$. Here, the restricted distance squared family is determined by the singular curve β. However, our aim is not only to recover invariants of β but to also organize the resulting contact strata as geometric data naturally attached to the pair (X, B). Furthermore, we obtain explicit formulas showing how the corresponding invariants are expressed in terms of the ambient cuspidal edge and the chosen source curve.

This point of view leads to two main conclusions: First, the lower contact strata admit a simple description attached to the pair; in particular, centres with at least second-order contact form a plane, while those with at least third-order contact form a line. Second, the fourth-order condition can be described in two complementary ways: one is derived from a scalar determined by the 4-jet of β, while the other is from an explicit adapted-normal-form expression involving coefficients of the cuspidal edge and the source curve.

The main results of the study are summarized below. Let $\kappa_{\mathrm{sing}}(\beta, 0)$, $\tau_{\mathrm{sing}}(\beta, 0)$, and $\mathbf{b}_{\mathrm{sing}}(\beta, 0)$ denote the cuspidal curvature, singular torsion, and singular binormal direction of the distinguished cusp $\beta = f \circ \alpha$. In Section 3, after studying the restricted distance squared family of the pair, we prove the main sphere-contact theorem; show that the centres with at least second-order contact form the singular normal plane, while those with at least

third-order contact form the singular binormal line; and prove that a unique centre with at least fourth-order contact exists if and only if $\tau_{\mathrm{sing}}(\beta,0) \neq 0$. In this case, the centre is

$$q_{\mathrm{osc}}(X,B) = \beta(0) + \frac{3}{\kappa_{\mathrm{sing}}(\beta,0)\,\tau_{\mathrm{sing}}(\beta,0)}\,\mathbf{b}_{\mathrm{sing}}(\beta,0).$$

In Section 3.3, the results are expressed in adapted normal form, and we determine the exact A_2- and A_3-strata and obtain a fifth-order criterion at the osculating centre in Section 4. In Section 5, we provide examples illustrating various cases, which yields explicit formulas for the lower strata, the fourth-order invariant Δ, and the singular torsion in terms of the coefficients of the cuspidal edge and the source curve. Finally, the conclusion and further directions are detailed in Section 6.

2. Cuspidal Edge–Cusp Pairs and Singular Invariants

In this section, we introduce the geometric setting and notation that will be used throughout the paper. We first recall the adapted normal form of a cuspidal edge and then explain how a source curve produces the distinguished cusp of the pair. Finally, we discuss the singular invariants and loci that will later describe the sphere-contact strata.

2.1. Cuspidal Edges and Adapted Normal Form

Let $f\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a smooth map-germ. We say that f has a cuspidal edge at the origin if it is $\mathcal{A}$-equivalent to $(u,v) \longmapsto (u,v^2,v^3)$.

We work in adapted coordinates on the source. A coordinate system (u,v) centred at the origin is labelled adapted if

(i) The u-axis is the singular set of f.
(ii) $\eta = \partial_v$ is a null vector field along the singular set.
(iii) The u-axis is the only singular locus near the origin.

In such coordinates, one has $f_v(u,v) = v\,\psi(u,v)$ for some smooth map-germ $\psi\colon (\mathbb{R}^2,0) \to \mathbb{R}^3$, and the vectors f_u and ψ are linearly independent near the origin.

We use the normal form below given in [7] (Theorem 3.1, Equation (3.4), p. 450).

Proposition 1. *Let $f : (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a map-germ with a cuspidal edge at the origin. Then, after a change of coordinates in the source and a Euclidean motion in the target, f can be written in the form*

$$f(u,v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{1}{2}v^2, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v),$$

where $b_{20} \geq 0$, $b_{03} \neq 0$, and

$$h(u,v) = \Big(0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v)\Big).$$

Here, h_1, h_2, h_3, and h_4 are smooth function-germs in u, and h_5 is a smooth function-germ in (u,v).

Remark 1. *For later use, write $h_5(u,v) = h_5(0,0) + \widetilde{h}_5(u,v)$, where $\widetilde{h}_5(0,0) = 0$, and set $b_{04} = 24h_5(0,0)$. Then,*

$$v^4h_5(u,v) = \frac{b_{04}}{24}v^4 + v^4\widetilde{h}_5(u,v).$$

Remark 2. *The coefficients in Proposition 1 describe the Euclidean geometry of the cuspidal edge. The coefficient a_{20} is its singular curvature, and b_{20} is the absolute value of its limiting normal curvature. With the normal-form orientation conventions, b_{03}, b_{12}, and b_{30} are, respectively,*

the surface cuspidal curvature, cusp-directional torsion, and edge inflectional curvature (see [7] (Theorem 4.4 and Section 5)).

Let κ_s and κ_ν denote the surface singular curvature and signed limiting normal curvature, with the unit normal chosen so that $\nu(0) = (0,0,1)$. Then,

$$a_{30} = \kappa_s'(0) - b_{20}b_{12}, \qquad b_{30} = \kappa_\nu'(0) + a_{20}b_{12},$$

where primes denote differentiation with respect to oriented arc length along the regular singular image $u \mapsto f(u,0)$.

When comparing normal-form representatives under source diffeomorphisms and Euclidean target isometries, a_{20} and b_{20} are unchanged. The magnitudes $|a_{30}|$, $|b_{30}|$, $|b_{12}|$, and $|b_{03}|$ are also invariant, but their signs depend on the remaining choices of orientation. For example, replacing v by $-v$ reverses b_{03}. The source change $(u,v) \mapsto (-u,v)$, together with the target reflection $(x,y,z) \mapsto (-x,y,z)$, reverses a_{30}, b_{30}, and b_{12}. Thus, signed coefficient values are understood relative to the chosen normal-form axes.

2.2. The Distinguished Cusp

Let $\alpha\colon (\mathbb{R},0) \to (\mathbb{R}^2,0), \alpha(s) = (u(s),v(s))$ be a smooth source curve-germ. After denoting its image under f by $\beta = f \circ \alpha$, the corresponding curve on the cuspidal edge is then $B = \operatorname{Im}\beta \subset X$.

Definition 1. *A cuspidal edge–cusp pair is a pair (X,B) where $X = f((\mathbb{R}^2,0)) \subset \mathbb{R}^3$ is a cuspidal edge germ and $B = \operatorname{Im}\beta$, with $\beta = f \circ \alpha$, for a smooth source curve-germ $\alpha\colon (\mathbb{R},0) \to (\mathbb{R}^2,0)$ such that β is a $(2,3)$-type cuspidal space curve, meaning that β is $\mathcal{A}$-equivalent to $s \longmapsto (0,s^2,s^3)$.*

We recall the standard criterion for such a $(2,3)$-type cusp in 3-space.

Proposition 2 ([2])**.** *Let $\gamma\colon (\mathbb{R},0) \to (\mathbb{R}^3,0)$ be a smooth curve-germ. Then, γ is $\mathcal{A}$-equivalent to $s \longmapsto (0,s^2,s^3)$ if and only if $\gamma'(0) = 0$, and $\gamma''(0) \times \gamma'''(0) \neq 0$.*

Applied to $\beta = f \circ \alpha$, this gives the following criterion in adapted coordinates.

Proposition 3. *Let $f\colon (\mathbb{R}^2,0) \to (\mathbb{R}^3,0)$ be a cuspidal edge in adapted coordinates, and let $\alpha(s) = (u(s),v(s))$ be a smooth source curve-germ. Then, $\beta = f \circ \alpha$ is a $(2,3)$-type cuspidal space curve if and only if $u'(0) = 0$, and $\beta''(0) \times \beta'''(0) \neq 0$. Whenever these conditions hold, $v'(0) \neq 0$. In particular, the source curve α is necessarily regular.*

Proof. In adapted coordinates, $f_v(0) = 0$ and $f_u(0) \neq 0$. Hence,

$$\beta'(0) = f_u(0)u'(0) + f_v(0)v'(0) = f_u(0)u'(0),$$

so $\beta'(0) = 0$ if and only if $u'(0) = 0$. By Proposition 2, the curve β is a $(2,3)$-type cusp if and only if $\beta'(0) = 0$ and $\beta''(0) \times \beta'''(0) \neq 0$. We have shown that $\beta'(0) = 0$ is equivalent to $u'(0) = 0$. Thus, these are precisely the two conditions in the statement.

If we assume that these conditions hold, the full second derivative formula is

$$\begin{aligned}\beta''(0) = {} & f_{uu}(0)\big(u'(0)\big)^2 + 2f_{uv}(0)u'(0)v'(0) + f_{vv}(0)\big(v'(0)\big)^2 \\ & + f_u(0)u''(0) + f_v(0)v''(0).\end{aligned}$$

Using $u'(0) = 0$ and $f_v(0) = 0$, this reduces to

$$\beta''(0) = f_u(0)u''(0) + f_{vv}(0)\big(v'(0)\big)^2.$$

If $v'(0) = 0$, then $\alpha'(0) = 0$, and the chain rule gives

$$\beta''(0) = f_u(0)u''(0), \qquad \beta'''(0) = f_u(0)u'''(0).$$

Both vectors are parallel to $f_u(0)$, so $\beta''(0) \times \beta'''(0) = 0$. Since this cross product is nonzero for a $(2,3)$-cusp, we must have $v'(0) \neq 0$. Hence, α is regular. □

Now, since $v'(0) \neq 0$, the function v is a local diffeomorphism at the origin. Hence, after a reparametrization of the source curve, we can assume that $v(s) = s$. Therefore, $\alpha(s) = (u(s), s)$ and $u(0) = u'(0) = 0$. After writing

$$u(s) = \frac{c_2}{2}s^2 + \frac{c_3}{6}s^3 + O(s^4),$$

we obtain the expanded formula below.

Corollary 1. *The distinguished cusp has the expansion*

$$\beta(s) = f(\alpha(s)) = \left(\frac{c_2}{2}s^2 + \frac{c_3}{6}s^3 + O(s^4),\ \frac{1}{2}s^2 + O(s^4),\ \frac{b_{03}}{6}s^3 + O(s^4)\right), \quad \textit{with} \quad b_{03} \neq 0.$$

2.3. Singular Invariants

We now introduce the singular invariants of the distinguished cusp using the normalizations in [13] (Appendix A, p. 217). The signed quantities depend on the orientation of the parameter, which is explained below.

By Proposition 2, a $(2,3)$-type cuspidal space curve γ is characterized by

$$\gamma'(0) = 0, \qquad \gamma''(0) \neq 0, \qquad \gamma''(0) \times \gamma'''(0) \neq 0.$$

Definition 2. *Let γ be a $(2,3)$-type cuspidal space curve. Its cuspidal curvature and singular torsion at the origin are*

$$\kappa_{\text{sing}}(\gamma, 0) = \frac{\|\gamma''(0) \times \gamma'''(0)\|}{\|\gamma''(0)\|^{5/2}}, \qquad \tau_{\text{sing}}(\gamma, 0) = \frac{\sqrt{\|\gamma''(0)\|}\ \det(\gamma''(0), \gamma'''(0), \gamma^{(4)}(0))}{\|\gamma''(0) \times \gamma'''(0)\|^2}.$$

Its singular binormal direction is

$$\mathbf{b}_{\text{sing}}(\gamma, 0) = \frac{\gamma''(0) \times \gamma'''(0)}{\|\gamma''(0) \times \gamma'''(0)\|}.$$

Here, $\sqrt{\|\gamma''(0)\|} = \|\gamma''(0)\|^{1/2}$ denotes the square root of the Euclidean norm, as similarly indicated in [13] (Appendix A, p. 217).

Remark 3. *Let $\widetilde{\gamma} = \gamma \circ \varphi$, where $\varphi(0) = 0$ and $\lambda = \varphi'(0) \neq 0$. Set*

$$a = \gamma''(0), \qquad b = \gamma'''(0), \qquad c = \gamma^{(4)}(0), \qquad \mu = \varphi''(0), \qquad \nu = \varphi'''(0).$$

Since $\gamma'(0) = 0$, the chain rule gives

$$\begin{aligned}
\widetilde{\gamma}''(0) &= \lambda^2 a, \\
\widetilde{\gamma}'''(0) &= \lambda^3 b + 3\lambda\mu a, \\
\widetilde{\gamma}^{(4)}(0) &= \lambda^4 c + 6\lambda^2\mu b + (3\mu^2 + 4\lambda\nu)a.
\end{aligned}$$

Consequently,

$$\widetilde{\gamma}''(0) \times \widetilde{\gamma}'''(0) = \lambda^5 (a \times b),$$
$$\det\big(\widetilde{\gamma}''(0), \widetilde{\gamma}'''(0), \widetilde{\gamma}^{(4)}(0)\big) = \lambda^9 \det(a, b, c).$$

Substitution into the defining formulas yields

$$\kappa_{\mathrm{sing}}(\widetilde{\gamma}, 0) = \kappa_{\mathrm{sing}}(\gamma, 0),$$
$$\tau_{\mathrm{sing}}(\widetilde{\gamma}, 0) = \operatorname{sgn}(\lambda)\, \tau_{\mathrm{sing}}(\gamma, 0),$$
$$\mathbf{b}_{\mathrm{sing}}(\widetilde{\gamma}, 0) = \operatorname{sgn}(\lambda)\, \mathbf{b}_{\mathrm{sing}}(\gamma, 0).$$

Thus, cuspidal curvature is invariant under every regular reparametrization; singular torsion and the singular binormal are invariant under orientation-preserving reparametrizations and both change sign under an orientation reversal. When $\tau_{\mathrm{sing}} \neq 0$, these signs cancel in $3\mathbf{b}_{\mathrm{sing}}/(\kappa_{\mathrm{sing}}\tau_{\mathrm{sing}})$. Hence, the condition $\tau_{\mathrm{sing}} \neq 0$, the osculating centre, and its radius are independent of the parametrization.

For the pair (X, B), we use the corresponding singular invariants

$$\kappa_{\mathrm{sing}}(\beta, 0), \qquad \tau_{\mathrm{sing}}(\beta, 0), \qquad \mathbf{b}_{\mathrm{sing}}(\beta, 0).$$

Also, we introduce two basic loci associated with the pair.

Definition 3. *Let (X, B) be a cuspidal edge–cusp pair, represented by $\beta = f \circ \alpha$. Set*

$$n_{(X,B)} := \beta''(0) \times \beta'''(0), \qquad \Delta := \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big).$$

Define

$$\Pi_{(X,B)} := \beta(0) + \beta''(0)^{\perp}$$

and

$$\ell_{(X,B)} := \beta(0) + \mathbb{R}\, n_{(X,B)}.$$

We label $\Pi_{(X,B)}$ as the singular normal plane and $\ell_{(X,B)}$ as the singular binormal line of the pair.

Here, $\beta''(0)^{\perp} = \{x \in \mathbb{R}^3 : \langle x, \beta''(0)\rangle = 0\}$ denotes the plane through the origin orthogonal to $\beta''(0)$.

Remark 4. *Since $n_{(X,B)} = \beta''(0) \times \beta'''(0)$, we have*

$$\Delta = \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big) = \langle n_{(X,B)}, \beta^{(4)}(0)\rangle.$$

Also, we have

$$\tau_{\mathrm{sing}}(\beta, 0) = \frac{\sqrt{\|\beta''(0)\|}\, \Delta}{\|n_{(X,B)}\|^2}.$$

Hence,

$$\Delta \neq 0 \quad \Longleftrightarrow \quad \tau_{\mathrm{sing}}(\beta, 0) \neq 0.$$

3. Restricted Distance Squared Family and the Main Contact Theorem

For $q \in \mathbb{R}^3$, consider the usual distance squared function

$$d_q(x) = \|x - q\|^2, \qquad x \in \mathbb{R}^3.$$

Sphere contact with the distinguished cusp is described by restricting this function to B. Since B is parametrized by $\beta(s) = f(\alpha(s))$, the restriction becomes a one-variable function in the curve parameter s:

$$D_{(X,B)}(s,q) = d_q(\beta(s)) = \|\beta(s) - q\|^2, \qquad q \in \mathbb{R}^3.$$

For a fixed q, we write

$$D_q(s) = D_{(X,B)}(s,q).$$

Before concluding the main result of this section, the derivative formulas below are required.

Lemma 1. *Let $\beta\colon (\mathbb{R},0) \to (\mathbb{R}^3,0)$ be a smooth curve-germ, and let $q \in \mathbb{R}^3$. Then, the derivatives of $D_q(s) = \|\beta(s) - q\|^2$ are given by*

$$\begin{aligned}
D_q'(s) &= 2\langle \beta(s) - q, \beta'(s)\rangle,\\
D_q''(s) &= 2\langle \beta'(s), \beta'(s)\rangle + 2\langle \beta(s) - q, \beta''(s)\rangle,\\
D_q'''(s) &= 6\langle \beta'(s), \beta''(s)\rangle + 2\langle \beta(s) - q, \beta'''(s)\rangle,\\
D_q^{(4)}(s) &= 6\|\beta''(s)\|^2 + 8\langle \beta'(s), \beta'''(s)\rangle + 2\langle \beta(s) - q, \beta^{(4)}(s)\rangle,\\
D_q^{(5)}(s) &= 20\langle \beta''(s), \beta'''(s)\rangle + 10\langle \beta'(s), \beta^{(4)}(s)\rangle + 2\langle \beta(s) - q, \beta^{(5)}(s)\rangle.
\end{aligned}$$

Proof. The formulas follow by differentiating

$$D_q(s) = \langle \beta(s) - q, \beta(s) - q\rangle$$

successively and applying the product rules for the Euclidean inner product. □

At the origin, the above formulas can be simplified because $\beta'(0) = 0$. In particular, the following is obtained:

Corollary 2. *If $\beta'(0) = 0$, then*

$$\begin{aligned}
D_q'(0) &= 0,\\
D_q''(0) &= 2\langle \beta(0) - q, \beta''(0)\rangle,\\
D_q'''(0) &= 2\langle \beta(0) - q, \beta'''(0)\rangle,\\
D_q^{(4)}(0) &= 6\|\beta''(0)\|^2 + 2\langle \beta(0) - q, \beta^{(4)}(0)\rangle,\\
D_q^{(5)}(0) &= 20\langle \beta''(0), \beta'''(0)\rangle + 2\langle \beta(0) - q, \beta^{(5)}(0)\rangle.
\end{aligned}$$

For a smooth function-germ $h\colon (\mathbb{R},0) \to \mathbb{R}$, set $\widetilde{h}(s) = h(s) - h(0)$. We say that h has an A_k-singularity, where $k \geq 1$, if $\widetilde{h}$ is $\mathcal{R}$-equivalent to $\pm s^{k+1}$ [2]. Equivalently,

$$h'(0) = \cdots = h^{(k)}(0) = 0, \qquad h^{(k+1)}(0) \neq 0.$$

Definition 4. *We use A_∞ to denote the case in which the normalized germ $\widetilde{h}$ is flat at the origin; that is, $h^{(j)}(0) = 0$ for every $j \geq 1$.*

For $m \geq 2$, we say that h has an $A_{\geq m}$-singularity if

$$h'(0) = h''(0) = \cdots = h^{(m)}(0) = 0.$$

Thus, $A_{\geq m}$ includes both the finite types A_k, with $k \geq m$, and the flat case A_∞.

For the restricted distance squared family, all statements about the singularity type of D_q refer to the normalized germ

$$H_q(s) = D_q(s) - D_q(0) = \|\beta(s) - q\|^2 - \|\beta(0) - q\|^2.$$

Since $H_q^{(j)}(0) = D_q^{(j)}(0)$ for every $j \geq 1$, the derivative criteria below apply directly to D_q.

3.1. Main Sphere-Contact Theorem

The derivative formulas above can now be used to obtain the main geometric result. We denote the unique centre of the osculating sphere of the pair by $q_{\mathrm{osc}}(X, B)$.

Theorem 1. *Let (X, B) be a cuspidal edge–cusp pair. Then, the restricted distance squared family D_q of the pair satisfies the following:*

(1) For every $q \in \mathbb{R}^3$, the germ D_q is singular at the origin.

(2) The set of centres for which D_q has an $A_{\geq 2}$-singularity is the singular normal plane

$$\Pi_{(X,B)} = \beta(0) + \beta''(0)^{\perp}.$$

(3) The set of centres for which D_q has an $A_{\geq 3}$-singularity is the singular binormal line

$$\ell_{(X,B)} = \beta(0) + \mathbb{R}\big(\beta''(0) \times \beta'''(0)\big).$$

(4) There exists a unique centre

$$q_{\mathrm{osc}}(X, B) \in \ell_{(X,B)}$$

for which D_q has an $A_{\geq 4}$-singularity if and only if $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$.

(5) When $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$, the unique fourth-order centre is

$$q_{\mathrm{osc}}(X, B) = \beta(0) + \frac{3\|\beta''(0)\|^2}{\Delta}\, n_{(X,B)} = \beta(0) + \frac{3}{\kappa_{\mathrm{sing}}(\beta, 0)\, \tau_{\mathrm{sing}}(\beta, 0)}\, \mathbf{b}_{\mathrm{sing}}(\beta, 0).$$

(6) When $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$, the corresponding sphere has radius

$$r_{\mathrm{osc}}(X, B) = \frac{3}{|\kappa_{\mathrm{sing}}(\beta, 0)\, \tau_{\mathrm{sing}}(\beta, 0)|}.$$

(7) If $\tau_{\mathrm{sing}}(\beta, 0) = 0$, then no point of $\ell_{(X,B)}$ gives an $A_{\geq 4}$-singularity.

Proof. Since B is a $(2, 3)$-type cuspidal space curve, we have

$$\beta'(0) = 0, \qquad \beta''(0) \neq 0, \qquad \beta''(0) \times \beta'''(0) \neq 0.$$

By Corollary 2, $D_q'(0) = 0$ for every $q \in \mathbb{R}^3$. This proves (1).

For (2), the germ D_q has an $A_{\geq 2}$-singularity if and only if $D_q''(0) = 0$. Using Corollary 2, this condition is equivalent to $\langle \beta(0) - q, \beta''(0) \rangle = 0$, or, equivalently, $\langle q - \beta(0), \beta''(0) \rangle = 0$. Thus, the set of such centres is

$$\beta(0) + \beta''(0)^{\perp} = \Pi_{(X,B)}.$$

This proves (2).

For (3), the condition for an $A_{\geq 3}$-singularity is $D_q''(0) = D_q'''(0) = 0$. Again by Corollary 2, this is equivalent to $\langle q - \beta(0), \beta''(0) \rangle = 0$, and $\langle q - \beta(0), \beta'''(0) \rangle = 0$. Since

$\beta''(0)$ and $\beta'''(0)$ are linearly independent, their common orthogonal complement is the line generated by $\beta''(0) \times \beta'''(0)$. Therefore, the $A_{\geq 3}$-centres are precisely

$$\ell_{(X,B)} = \beta(0) + \mathbb{R}\big(\beta''(0) \times \beta'''(0)\big).$$

This proves (3).

Now, set $a = \beta''(0), b = \beta'''(0)$, and $n = n_{(X,B)} = a \times b$. If $q \in \ell_{(X,B)}$, then we may write $q = \beta(0) + t\,n, t \in \mathbb{R}$. By (3), such a point already gives an $A_{\geq 3}$-singularity. Hence, the remaining condition for an $A_{\geq 4}$-singularity is $D_q^{(4)}(0) = 0$. From Corollary 2,

$$D_q^{(4)}(0) = 6\|a\|^2 + 2\langle \beta(0) - q, \beta^{(4)}(0)\rangle.$$

Since $\beta(0) - q = -t\,n$, we obtain

$$D_q^{(4)}(0) = 6\|a\|^2 - 2t\langle n, \beta^{(4)}(0)\rangle.$$

At $t = 0$, the centre is $q = \beta(0)$ and $D_q^{(4)}(0) = 6\|a\|^2 > 0$. The first three derivatives vanish, so the normalized germ has the exact type A_3 and the corresponding radius is zero (see Remark 6). However,

$$\langle n, \beta^{(4)}(0)\rangle = \det(a, b, \beta^{(4)}(0)) = \Delta.$$

Therefore, the fourth-order condition is

$$6\|a\|^2 - 2t\,\Delta = 0.$$

If $\Delta \neq 0$, this equation has the unique solution $t = \frac{3\|a\|^2}{\Delta}$. As a result, there is a unique centre on $\ell_{(X,B)}$ giving an $A_{\geq 4}$-singularity, namely

$$q_{\mathrm{osc}}(X, B) = \beta(0) + \frac{3\|a\|^2}{\Delta}\, n.$$

If $\Delta = 0$, then the fourth-order condition becomes $6\|a\|^2 = 0$, though this is impossible because $a = \beta''(0) \neq 0$. Thus, no point of $\ell_{(X,B)}$ gives an $A_{\geq 4}$-singularity in this case. On the other hand,

$$\tau_{\mathrm{sing}}(\beta, 0) = \frac{\sqrt{\|a\|}\,\Delta}{\|n\|^2}.$$

Since $a \neq 0$ and $n = a \times b \neq 0$, we have

$$\Delta \neq 0 \quad \Longleftrightarrow \quad \tau_{\mathrm{sing}}(\beta, 0) \neq 0.$$

This proves (4) and (7).

The centre must be rewritten in terms of the singular invariants. By definition,

$$\kappa_{\mathrm{sing}}(\beta, 0) = \frac{\|n\|}{\|a\|^{5/2}}, \qquad \tau_{\mathrm{sing}}(\beta, 0) = \frac{\sqrt{\|a\|}\,\Delta}{\|n\|^2}.$$

Hence,

$$\kappa_{\mathrm{sing}}(\beta, 0)\,\tau_{\mathrm{sing}}(\beta, 0) = \frac{\Delta}{\|a\|^2\|n\|}.$$

Therefore,

$$\frac{3\|a\|^2}{\Delta}\, n = \frac{3}{\kappa_{\mathrm{sing}}(\beta, 0)\,\tau_{\mathrm{sing}}(\beta, 0)}\, \frac{n}{\|n\|} = \frac{3}{\kappa_{\mathrm{sing}}(\beta, 0)\,\tau_{\mathrm{sing}}(\beta, 0)}\, \mathbf{b}_{\mathrm{sing}}(\beta, 0).$$

Thus, when $\tau_{\mathrm{sing}}(\beta,0)\neq 0$,

$$q_{\mathrm{osc}}(X,B)=\beta(0)+\frac{3\|\beta''(0)\|^2}{\Delta}\,n_{(X,B)}=\beta(0)+\frac{3}{\kappa_{\mathrm{sing}}(\beta,0)\,\tau_{\mathrm{sing}}(\beta,0)}\,\mathbf{b}_{\mathrm{sing}}(\beta,0).$$

This proves (5).

Finally, the radius of the corresponding sphere is

$$r_{\mathrm{osc}}(X,B)=\|q_{\mathrm{osc}}(X,B)-\beta(0)\|.$$

Using the formula above, we obtain

$$r_{\mathrm{osc}}(X,B)=\left|\frac{3}{\kappa_{\mathrm{sing}}(\beta,0)\,\tau_{\mathrm{sing}}(\beta,0)}\right|=\frac{3}{|\kappa_{\mathrm{sing}}(\beta,0)\,\tau_{\mathrm{sing}}(\beta,0)|}.$$

This proves (6), and the proof is complete. □

Remark 5. *Let*

$$\ell(s)=\int_0^s\|\beta'(u)\|\,du$$

be the signed arc length of the distinguished cusp. The half-arclength parameter $t=\mathrm{sgn}(s)\sqrt{|\ell(s)|}$ *is an orientation-preserving smooth local parameter. Using* t *as the parameter of* β*, we have* $d\ell/dt=\|\beta'(t)\|=2|t|$*. On the regular part, let* $\kappa(t)$*,* $\tau(t)$*,* $\mathbf{N}(t)$*, and* $\mathbf{B}(t)$ *denote the classical curvature, torsion, principal normal, and binormal. The functions* $\widehat{\kappa}(t)=|t|\kappa(t)$ *and* $\widehat{\tau}(t)=t\tau(t)$ *extend smoothly to* $t=0$*, with*

$$\widehat{\kappa}(0)=\frac{\kappa_{\mathrm{sing}}(\beta,0)}{2\sqrt{2}},\qquad\widehat{\tau}(0)=\frac{2\tau_{\mathrm{sing}}(\beta,0)}{3\sqrt{2}};$$

see [13] (Appendix A, Fact A.1 and Proposition A.2).

Assume $\tau_{\mathrm{sing}}(\beta,0)\neq 0$*. Then,* $\tau(t)\neq 0$ *for a sufficiently small* $t\neq 0$*, and the classical osculating centre is*

$$q_{\mathrm{reg}}(t)=\beta(t)+\rho(t)\mathbf{N}(t)+\frac{1}{\tau(t)}\frac{d\rho}{d\ell}(t)\mathbf{B}(t),\qquad\rho(t)=\frac{1}{\kappa(t)}.\tag{1}$$

Since $\rho(t)=|t|/\widehat{\kappa}(t)$ *and* $d/d\ell=(2|t|)^{-1}d/dt$*, we obtain*

$$\frac{1}{\tau(t)}\frac{d\rho}{d\ell}(t)=\frac{\widehat{\kappa}(t)-t\widehat{\kappa}'(t)}{2\widehat{\kappa}(t)^2\widehat{\tau}(t)}.\tag{2}$$

Here, $\widehat{\kappa}'$ *denotes differentiation with respect to* t*. As* t *tends to zero, the right-hand side of* (2) *tends to* $3/(\kappa_{\mathrm{sing}}(\beta,0)\tau_{\mathrm{sing}}(\beta,0))$*. Moreover,* $\rho(t)\mathbf{N}(t)$ *tends to zero, and the Taylor expansion shows that* $\mathbf{B}(t)$ *tends to* $\mathbf{b}_{\mathrm{sing}}(\beta,0)$*. Consequently,* (1) *results in*

$$\lim_{t\to 0^{\pm}}q_{\mathrm{reg}}(t)=\beta(0)+\frac{3}{\kappa_{\mathrm{sing}}(\beta,0)\tau_{\mathrm{sing}}(\beta,0)}\mathbf{b}_{\mathrm{sing}}(\beta,0)=q_{\mathrm{osc}}(X,B).\tag{3}$$

Thus, (3) *shows that the centre in Theorem 1(5) is the common limit of the classical osculating centres on the two regular branches of the cusp.*

3.2. Sphere Contact Along the Distinguished Cusp

Thus far, we have described the restricted distance squared function D_q in terms of its singularity type, but the corresponding geometric meaning will now be explained.

For $q \neq \beta(0)$, let $S(q,r) = \{x \in \mathbb{R}^3 : \|x-q\|^2 = r^2\}$ be the sphere centred at q and passing through $\beta(0)$. Thus, $r = \|q-\beta(0)\| > 0$. Then,

$$D_q(s) - r^2 = \|\beta(s) - q\|^2 - \|\beta(0) - q\|^2$$

measures the contact of $S(q,r)$ with the curve B at the origin.

Definition 5. *We say that $S(q,r)$ has contact in the order of at least k with the pair (X,B) along B at the origin if $j_0^k(D_q - r^2) = 0$. Equivalently, $D_q^{(j)}(0) = 0, (1 \leq j \leq k)$. If, in addition, $D_q^{(k+1)}(0) \neq 0$, then the contact is said to be exactly of order k.*

Proposition 4. *Let (X,B) be a cuspidal edge–cusp pair, and assume that $q \neq \beta(0)$. Then, for $k \geq 2$, the germ D_q has an A_k-singularity if and only if the sphere centred at q has exactly k-th order contact with the pair along B. Equivalently,*

$$D_q'(0) = D_q''(0) = \cdots = D_q^{(k)}(0) = 0, \qquad D_q^{(k+1)}(0) \neq 0.$$

In particular, the following is true:

(i) D_q has an A_2-singularity if and only if $S(q,r)$ has exactly second-order contact with the pair along B.

(ii) D_q has an A_3-singularity if and only if $S(q,r)$ has exactly third-order contact with the pair along B.

(iii) D_q has an A_4-singularity if and only if $S(q,r)$ has exactly fourth-order contact with the pair along B.

Proof. Set $h_q(s) = D_q(s) - D_q(0)$. Since $D_q(0) = \|\beta(0) - q\|^2 = r^2$, we have $h_q(s) = D_q(s) - r^2$, so h_q measures the contact between $S(q,r)$ and the distinguished cusp.

By definition, D_q has an A_k-singularity at the origin if its normalized germ h_q is $\mathcal{R}$-equivalent to $\pm s^{k+1}$. For a one-variable germ, this is equivalent to

$$h_q'(0) = h_q''(0) = \cdots = h_q^{(k)}(0) = 0, \qquad h_q^{(k+1)}(0) \neq 0.$$

Since $h_q^{(j)}(0) = D_q^{(j)}(0)$, for all $j \geq 1$, this is precisely the condition under which $S(q,r)$ has exactly k-th order contact with B at the origin. The cases $k = 2, 3, 4$ follow immediately. □

Remark 6. *The point $p = \beta(0)$ lies on $\ell_{(X,B)}$, corresponding to $t = 0$. For this centre,*

$$H_p(s) = \|\beta(s) - p\|^2 = \frac{\|\beta''(0)\|^2}{4} s^4 + O(s^5).$$

Since $\beta''(0) \neq 0$, the germ H_p has the exact type A_3, and because the associated radius is zero, the corresponding level set consists only of the point p.

We include p in the singularity strata of the distance squared family, but for spheres of positive radius, we exclude this point. The centres with contact in the order of at least two and three are designated as $\Pi_{(X,B)} \setminus \{p\}$ and $\ell_{(X,B)} \setminus \{p\}$, respectively.

3.3. Adapted Normal Form and the Pair Viewpoint

The above results were stated in terms of the image cusp

$$\beta = f \circ \alpha.$$

We can now rewrite them using the adapted normal form of the pair (X, B), but the contact theory itself is unchanged. We aim to show how the invariants of β are expressed through the coefficients of the cuspidal edge f and the source curve α.

3.3.1. Normal-Form Description of the Lower Strata

Assume that f is written in adapted normal form. After reparametrizing the source curve, we can have $\alpha(s) = (u(s), s)$. We write the Taylor expansion of u as

$$u(s) = \frac{c_2}{2}s^2 + \frac{c_3}{6}s^3 + \frac{c_4}{24}s^4 + O(s^5), \quad \text{where} \quad c_i \in \mathbb{R}.$$

Substituting $\alpha(s)$ into f, we obtain

$$\beta(s) = f(\alpha(s)) = \Big(\tfrac{c_2}{2}s^2 + \tfrac{c_3}{6}s^3 + \tfrac{c_4}{24}s^4 + O(s^5),\ \tfrac{1}{2}s^2 + \tfrac{y_4}{24}s^4 + O(s^5),\ \tfrac{b_{03}}{6}s^3 + \tfrac{z_4}{24}s^4 + O(s^5)\Big),$$

where $y_4 = 3a_{20}c_2^2$, $z_4 = 3b_{20}c_2^2 + 6b_{12}c_2 + b_{04}$, and $b_{03} \neq 0$. Hence, we obtain

$$\beta''(0) = (c_2, 1, 0), \tag{4}$$
$$\beta'''(0) = (c_3, 0, b_{03}), \tag{5}$$
$$\beta^{(4)}(0) = (c_4, y_4, z_4). \tag{6}$$

Proposition 5. *Assume $\beta(0) = 0$, and let $q = (x_0, y_0, z_0) \in \mathbb{R}^3$. Then, the following hold:*

(1) $q \in \Pi_{(X,B)}$ if and only if $c_2x_0 + y_0 = 0$.

(2) The singular binormal line is

$$\ell_{(X,B)} = \{t(\beta''(0) \times \beta'''(0)) : t \in \mathbb{R}\} = \{t\,(b_{03}, -c_2b_{03}, -c_3) : t \in \mathbb{R}\}.$$

(3) The cuspidal curvature of the distinguished cusp β is

$$\kappa_{\text{sing}}(\beta, 0) = \frac{\sqrt{b_{03}^2(1 + c_2^2) + c_3^2}}{(1 + c_2^2)^{5/4}}.$$

(4) The singular torsion of the distinguished cusp is

$$\tau_{\text{sing}}(\beta, 0) = \frac{(1 + c_2^2)^{1/4}\,\Delta}{b_{03}^2(1 + c_2^2) + c_3^2},$$

where $\Delta = b_{03}c_4 - c_2b_{03}y_4 - c_3z_4$.

Proof. Since $\beta(0) = 0$, the equation of $\Pi_{(X,B)}$ is $\langle q, \beta''(0)\rangle = 0$. Using Equation (4), we have $\beta''(0) = (c_2, 1, 0)$, and, subsequently, $\langle (x_0, y_0, z_0), (c_2, 1, 0)\rangle = c_2x_0 + y_0$. Therefore, $q \in \Pi_{(X,B)}$ if and only if $c_2x_0 + y_0 = 0$, which proves (1).

Using Equations (4) and (5), we obtain $\beta''(0) \times \beta'''(0) = (c_2, 1, 0) \times (c_3, 0, b_{03}) = (b_{03}, -c_2b_{03}, -c_3)$. Since $b_{03} \neq 0$, this vector is nonzero. Therefore, it spans the singular binormal line, and we obtain

$$\ell_{(X,B)} = \{t\,(b_{03}, -c_2b_{03}, -c_3) : t \in \mathbb{R}\}.$$

This proves (2).

For (3), we compute the two norms:

$$\|\beta''(0)\| = \sqrt{c_2^2 + 1} = \sqrt{1 + c_2^2},$$
$$\|\beta''(0) \times \beta'''(0)\| = \sqrt{b_{03}^2 + c_2^2 b_{03}^2 + c_3^2} = \sqrt{b_{03}^2(1 + c_2^2) + c_3^2}.$$

Substituting these expressions into the definition

$$\kappa_{\text{sing}}(\beta, 0) = \frac{\|\beta''(0) \times \beta'''(0)\|}{\|\beta''(0)\|^{5/2}}$$

gives

$$\kappa_{\text{sing}}(\beta, 0) = \frac{\sqrt{b_{03}^2(1 + c_2^2) + c_3^2}}{(1 + c_2^2)^{5/4}},$$

as required.

Finally, using Equations (4), (5), and (6), we have

$$\begin{aligned}\Delta &= \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big) \\ &= \Big\langle \beta''(0) \times \beta'''(0), \beta^{(4)}(0) \Big\rangle \\ &= \langle (b_{03}, -c_2 b_{03}, -c_3), (c_4, y_4, z_4) \rangle \\ &= b_{03} c_4 - c_2 b_{03} y_4 - c_3 z_4.\end{aligned}$$

Moreover,

$$\sqrt{\|\beta''(0)\|} = (1 + c_2^2)^{1/4} \quad \text{and} \quad \|\beta''(0) \times \beta'''(0)\|^2 = b_{03}^2(1 + c_2^2) + c_3^2.$$

Substituting these expressions and the above formula for Δ into the definition

$$\tau_{\text{sing}}(\beta, 0) = \frac{\sqrt{\|\beta''(0)\|}\, \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big)}{\|\beta''(0) \times \beta'''(0)\|^2}$$

gives

$$\tau_{\text{sing}}(\beta, 0) = \frac{(1 + c_2^2)^{1/4}\, \Delta}{b_{03}^2(1 + c_2^2) + c_3^2}.$$

This proves (4). □

Remark 7. *Along the singular binormal line, fourth-order contact is controlled by the scalar Δ. This is the first point where the 4-jet of the distinguished cusp enters the contact problem. It also connects the intrinsic description from Section 3 with the adapted normal form of the pair.*

Proposition 6. *Assume that $q \in \ell_{(X,B)}$ and $q = t\,(b_{03}, -c_2 b_{03}, -c_3)$, with $t \in \mathbb{R}$.*

Then, D_q has an $A_{\geq 4}$-singularity if and only if

$$3(1 + c_2^2) - t\,\Delta = 0,$$

where $\Delta = b_{03} c_4 - c_2 b_{03} y_4 - c_3 z_4$.

Proof. Since $\beta(0) = 0$, the fourth-order condition from Theorem 1 is

$$6\|\beta''(0)\|^2 - 2\langle q, \beta^{(4)}(0) \rangle = 0.$$

Using Equation (4), we have $\|\beta''(0)\|^2 = 1 + c_2^2$. Using Equation (6) and the expression for q, we obtain $\langle q, \beta^{(4)}(0)\rangle = t\,\Delta$. Therefore, the fourth-order condition becomes

$$6(1 + c_2^2) - 2t\,\Delta = 0,$$

or, equivalently,

$$3(1 + c_2^2) - t\,\Delta = 0.$$

This proves the result. □

Corollary 3. *Under the assumptions of Proposition 6, the following hold:*

(i) *If $\Delta \neq 0$, then there exists a unique point q on $\ell_{(X,B)}$ for which D_q has an $A_{\geq 4}$-singularity. In particular,*

$$q_{\mathrm{osc}}(X, B) = \frac{3(1 + c_2^2)}{\Delta}\,(b_{03}, -c_2 b_{03}, -c_3).$$

(ii) *If $\Delta = 0$, then no point of $\ell_{(X,B)}$ gives an $A_{\geq 4}$-singularity.*

Proof. By Proposition 6, the condition for an $A_{\geq 4}$-singularity is

$$3(1 + c_2^2) - t\,\Delta = 0. \tag{7}$$

If $\Delta \neq 0$, this equation has the unique solution $t = \frac{3(1+c_2^2)}{\Delta}$, which gives the stated formula for $q_{\mathrm{osc}}(X, B)$. If $\Delta = 0$, then Equation (7) becomes $3(1 + c_2^2) = 0$, which is impossible. Consequently, no such point exists. □

3.3.2. Adapted Normal Form and the Contact Invariants

The quantity

$$\Delta = \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big)$$

admits an explicit expression for the adapted normal form. Now, we are able to clarify its geometric meaning; more precisely, we distinguish the information determined by the image cusp $\beta = f \circ \alpha$ from the particular way this information is represented in adapted coordinates for the pair (X, B).

Remark 8. *The singular normal plane $\Pi_{(X,B)}$, the singular binormal line $\ell_{(X,B)}$, and the cuspidal curvature $\kappa_{\mathrm{sing}}(\beta, 0)$ are determined by $\beta(0), \beta''(0)$, and $\beta'''(0)$. Equivalently, they depend only on the 3-jet $j^3\beta(0)$. By contrast, the invariant*

$$\Delta = \det\big(\beta''(0), \beta'''(0), \beta^{(4)}(0)\big)$$

involves the fourth derivative of β.

Remark 9. *When $\Delta \neq 0$, Theorem 1 gives the osculating centre in adapted normal form as*

$$q_{\mathrm{osc}}(X, B) = \frac{3(1 + c_2^2)}{\Delta}\,(b_{03}, -c_2 b_{03}, -c_3).$$

The existence and position of the osculating centre depend only on $j^4\beta(0)$.

Corollary 4. *Let (X_1, B_1) and (X_2, B_2) be cuspidal edge–cusp pairs.*

(i) *If the distinguished cusps have the same 3-jet at the origin, then the two pairs have the same singular normal plane, singular binormal line, and cuspidal curvature.*

(ii) *If the distinguished cusps have the same* 4*-jet at the origin, then the two pairs also have the same singular torsion and osculating centre, whenever it exists.*

Remark 10. *When the lower-order data $\Pi_{(X,B)}$ and $\ell_{(X,B)}$ are fixed, the fourth-order behaviour along the singular binormal line is governed by the scalar Δ. In adapted normal form, the fourth-order centre appears at the parameter value*

$$t = \frac{3(1+c_2^2)}{\Delta}.$$

Thus, Δ determines both the existence of the osculating centre and its position on the singular binormal line.

4. Exact Strata and a Fifth-Order Criterion

The main theorem describes a nested family of loci where the contact order increases. In this section, we make this stratification more precise by first identifying the exact A_2- and A_3-strata inside the larger contact loci. We then derive the first condition beyond fourth-order contact, namely the criterion for the osculating centre, to achieve an $A_{\geq 5}$-singularity.

Proposition 7. *Let (X, B) be a cuspidal edge–cusp pair. Then, the following hold:*

(1) The set of centres giving an exact A_2-singularity is $\Pi_{(X,B)} \setminus \ell_{(X,B)}$.
(2) If $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$, then the set of centres giving an exact A_3-singularity is $\ell_{(X,B)} \setminus \{q_{\mathrm{osc}}(X, B)\}$.
(3) If $\tau_{\mathrm{sing}}(\beta, 0) = 0$, then every point of $\ell_{(X,B)}$ gives an exact A_3-singularity.
(4) If $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$, then the unique centre giving an $A_{\geq 4}$-singularity is $q_{\mathrm{osc}}(X, B)$. This centre gives an exact A_4-singularity if and only if $D^{(5)}_{q_{\mathrm{osc}}}(0) \neq 0$. If instead $D^{(5)}_{q_{\mathrm{osc}}}(0) = 0$, then the normalized germ either has finite type A_k for some $k \geq 5$ or is flat (A_∞). Both possibilities are included in the notation $A_{\geq 5}$.

Proof. By Theorem 1, the $A_{\geq 2}$-locus is exactly $\Pi_{(X,B)}$, while the $A_{\geq 3}$-locus is exactly $\ell_{(X,B)}$. Therefore, the centres with the exact type A_2 are precisely $\Pi_{(X,B)} \setminus \ell_{(X,B)}$. This proves (1).

Assume now that $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$. Then, according to Theorem 1, a unique point $q_{\mathrm{osc}}(X, B) \in \ell_{(X,B)}$ is obtained for which the contact order is at least 4. As a result, every other point of $\ell_{(X,B)}$ produces an $A_{\geq 3}$-singularity but not an $A_{\geq 4}$-singularity. Thus, every point of $\ell_{(X,B)} \setminus \{q_{\mathrm{osc}}(X, B)\}$ has the exact type A_3. This proves (2).

If $\tau_{\mathrm{sing}}(\beta, 0) = 0$, then Theorem 1 shows that no point of $\ell_{(X,B)}$ gives an $A_{\geq 4}$-singularity. Since $\ell_{(X,B)}$ is the whole $A_{\geq 3}$-locus, every point of $\ell_{(X,B)}$ has the exact type A_3. This proves (3).

Finally, assume again that $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$. Then, $q_{\mathrm{osc}}(X, B)$ is the only centre that can produce an $A_{\geq 4}$-singularity. The next derivative must be examined to decide whether the type is exactly A_4 or at least A_5. By Proposition 4, the type is exactly A_4 if and only if $D^{(5)}_{q_{\mathrm{osc}}}(0) \neq 0$. If this derivative vanishes, then the singularity is of type $A_{\geq 5}$. This proves (4). □

Remark 11. *In the generic case $\tau_{\mathrm{sing}}(\beta, 0) \neq 0$, the centre space decomposes as*

$$\underbrace{\mathbb{R}^3 \setminus \Pi_{(X,B)}}_{\text{not } A_{\geq 2}}, \quad \underbrace{\Pi_{(X,B)} \setminus \ell_{(X,B)}}_{A_2}, \quad \underbrace{\ell_{(X,B)} \setminus \{q_{\mathrm{osc}}(X, B)\}}_{A_3}, \quad \underbrace{\{q_{\mathrm{osc}}(X, B)\}}_{A_{\geq 4}}.$$

Figure 1 illustrates this stratification of the centre space.

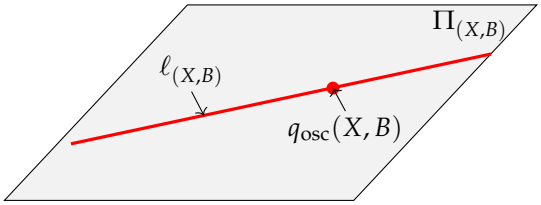


Figure 1. Centre stratification in the generic case $\tau_{\mathrm{sing}}(\beta,0)\neq 0$: The singular normal plane $\Pi_{(X,B)}$, the singular binormal line $\ell_{(X,B)}$, and the unique osculating centre $q_{\mathrm{osc}}(X,B)$.

We subsequently derive the fifth-order condition at the osculating centre.

Proposition 8. *Assume $\tau_{\mathrm{sing}}(\beta,0)\neq 0$. Then, $D_{q_{\mathrm{osc}}}$ has an $A_{\geq 5}$-singularity if and only if*

$$10\,\Delta\,\langle\beta''(0),\beta'''(0)\rangle - 3\|\beta''(0)\|^2\,\langle n_{(X,B)},\beta^{(5)}(0)\rangle = 0.$$

Proof. By Corollary 2, we have

$$D_q^{(5)}(0) = 20\langle\beta''(0),\beta'''(0)\rangle + 2\langle\beta(0)-q,\beta^{(5)}(0)\rangle.$$

At the osculating centre, we have

$$q = q_{\mathrm{osc}}(X,B) = \beta(0) + \frac{3\|\beta''(0)\|^2}{\Delta}\,n_{(X,B)}.$$

Hence,

$$\beta(0) - q = -\frac{3\|\beta''(0)\|^2}{\Delta}\,n_{(X,B)}.$$

After substituting this into the expression for $D_q^{(5)}(0)$, we obtain

$$D_{q_{\mathrm{osc}}}^{(5)}(0) = 20\langle\beta''(0),\beta'''(0)\rangle - \frac{6\|\beta''(0)\|^2}{\Delta}\langle n_{(X,B)},\beta^{(5)}(0)\rangle.$$

Since $\tau_{\mathrm{sing}}(\beta,0)\neq 0$, Theorem 1 gives $\Delta\neq 0$. Therefore, $D_{q_{\mathrm{osc}}}^{(5)}(0)=0$ if and only if

$$20\,\Delta\,\langle\beta''(0),\beta'''(0)\rangle - 6\|\beta''(0)\|^2\,\langle n_{(X,B)},\beta^{(5)}(0)\rangle = 0.$$

When dividing by 2, we obtain

$$10\,\Delta\,\langle\beta''(0),\beta'''(0)\rangle - 3\|\beta''(0)\|^2\,\langle n_{(X,B)},\beta^{(5)}(0)\rangle = 0.$$

This proves the result. □

Remark 12. *The fifth-order criterion includes the flat case and characterizes contact in the order of at least five, without requiring the normalized distance squared germ to have a finite A_k-type.*

5. Examples

We conclude with three situations illustrating different aspects of the contact geometry of cuspidal edge–cusp pairs: The first describes the degenerate case in which the singular torsion vanishes and no fourth-order centre exists. The second shows that, even for a fixed ambient cuspidal edge, the lower-order contact data do not determine the fourth-order behaviour. The third shows that the same 4-jet of the distinguished cusp may arise from different pair presentations. In such cases, the adapted-normal-form expression of the fourth-order invariant may be decomposed differently.

5.1. Degenerate Case: Absence of a Fourth-Order Centre

Consider

$$f(u,v) = \left(u, \frac{1}{2}v^2, \frac{1}{6}v^3\right), \qquad \alpha(s) = \left(\frac{1}{2}s^2, s\right).$$

Then,

$$\beta(s) = f(\alpha(s)) = \left(\frac{1}{2}s^2, \frac{1}{2}s^2, \frac{1}{6}s^3\right).$$

Hence,

$$\beta''(0) = (1,1,0), \qquad \beta'''(0) = (0,0,1),$$

and therefore,

$$n_{(X,B)} = \beta''(0) \times \beta'''(0) = (1,-1,0).$$

It follows that

$$\Pi_{(X,B)} = \{(x,y,z) \in \mathbb{R}^3 : \ x+y=0\}, \qquad \ell_{(X,B)} = \{t(1,-1,0) : \ t \in \mathbb{R}\}.$$

Moreover, $\beta^{(4)}(0) = 0$, and, therefore,

$$\Delta = \det\left(\beta''(0), \beta'''(0), \beta^{(4)}(0)\right) = 0.$$

Thus, $\tau_{\text{sing}}(\beta, 0) = 0$, so $\beta^{(4)}(0)$ lies in the plane spanned by $\beta''(0)$ and $\beta'''(0)$. As a result, the cusp has no fourth-order component transverse to this plane. Consequently, the pair admits no fourth-order centre. Equivalently, every point of $\ell_{(X,B)}$ gives an exact A_3-singularity. Here,"exact A_3" means that, for every $q \in \ell_{(X,B)}$, the squared distance function

$$d_q(s) = \|\beta(s) - q\|^2$$

has an A_3-singularity at $s = 0$ but not an $A_{\geq 4}$-singularity. Thus, all points of $\ell_{(X,B)}$ exhibit third-order contact, and no point on this line shows fourth-order contact.

This example corresponds to the degenerate case in Theorem 1. The singular binormal line is still well defined since it depends only on $\beta''(0)$ and $\beta'''(0)$. However, no osculating centre is selected on this line because the fourth-order invariant Δ vanishes.

5.2. A Fixed Cuspidal Edge with Different Fourth-Order Behaviour

Consider the cuspidal edge

$$f(u,v) = \left(u, \frac{1}{2}u^2 + \frac{1}{2}v^2, u^2 + \frac{1}{2}uv^2 + \frac{1}{6}v^3 + \frac{1}{12}v^4\right).$$

Here, $a_{20} = 1, b_{20} = 2, b_{12} = 1, b_{03} = 1$, and $b_{04} = 2$. Choose two distinguished source curves on the same edge:

$$\alpha_1(s) = \left(\frac{1}{2}s^2 + \frac{1}{6}s^3 + \frac{1}{24}s^4, s\right), \qquad \alpha_2(s) = \left(\frac{1}{2}s^2 + \frac{1}{6}s^3 + \frac{17}{24}s^4, s\right).$$

For both curves, $c_2 = 1$ and $c_3 = 1$. Therefore,

$$\beta_1''(0) = \beta_2''(0) = (1,1,0), \qquad \beta_1'''(0) = \beta_2'''(0) = (1,0,1).$$

Thus, the two pairs have the same lower-order contact data. In particular,

$$\Pi_{(X,B_i)} = \{(x,y,z) \in \mathbb{R}^3 : \ x+y=0\},$$

and

$$\ell_{(X,B_i)} = \{t(1,-1,-1) : t \in \mathbb{R}\}, \qquad i = 1,2.$$

They also have the same cuspidal curvature.

The difference appears at the fourth order. For α_1, we have $c_4 = 1$. By Proposition 5,

$$\Delta_1 = 1 - 3 - (6+6+2) = -16.$$

Hence, (X, B_1) admits a unique fourth-order centre, namely

$$q_{\mathrm{osc}}(X, B_1) = \frac{3(1+1)}{-16}(1,-1,-1) = \left(-\frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right).$$

For α_2, we have $c_4 = 17$. Therefore,

$$\Delta_2 = 17 - 3 - (6+6+2) = 0.$$

Thus, (X, B_2) admits no fourth-order centre.

This example shows that the lower-order contact data $\Pi_{(X,B)}$, $\ell_{(X,B)}$, and $\kappa_{\mathrm{sing}}(\beta, 0)$ do not determine the fourth-order behaviour, even when the ambient cuspidal edge is fixed. Figure 2 illustrates this distinction for the two pairs (X, B_1) and (X, B_2).

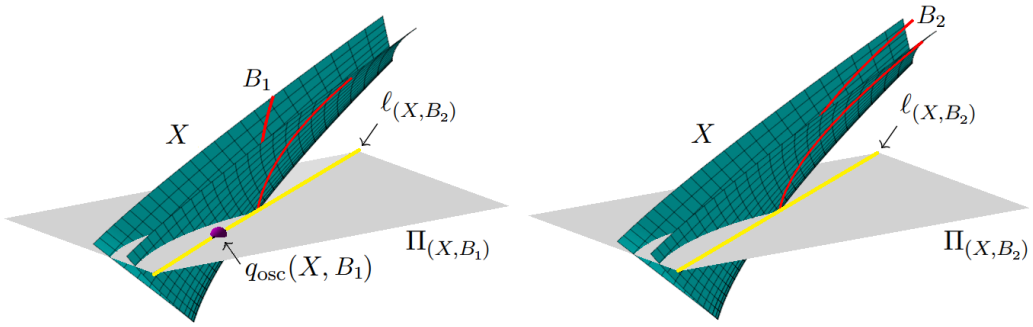


Figure 2. Both figures show the same cuspidal edge surface X. The left-hand figure shows the cusp curve B_1 determined by the source curve α_1, whereas the right-hand figure shows the cusp curve B_2 determined by α_2. Both have the same lower-order contact data, but only (X, B_1) admits a fourth-order centre.

5.3. Same Cusp 4-jet and Different Pair Presentations

Now, we are able to compare two pair presentations that demonstrate the same 4-jet of the distinguished cusp. Let

$$\alpha(s) = \left(\frac{1}{2}s^2 + \frac{1}{6}s^3 + \frac{1}{24}s^4, s\right),$$

and consider the cuspidal edges

$$f_A(u,v) = \left(u, \frac{1}{2}u^2 + \frac{1}{2}v^2, u^2 + \frac{1}{2}uv^2 + \frac{1}{6}v^3 + \frac{1}{12}v^4\right)$$

and

$$f_B(u,v) = \left(u, \frac{1}{2}u^2 + \frac{1}{2}v^2, uv^2 + \frac{1}{6}v^3 + \frac{1}{12}v^4\right).$$

For f_A, the relevant coefficients are

$$(a_{20}, b_{20}, b_{12}, b_{03}, b_{04}) = (1,2,1,1,2),$$

whereas for f_B, they are

$$(a_{20}, b_{20}, b_{12}, b_{03}, b_{04}) = (1, 0, 2, 1, 2).$$

In both cases, $c_2 = c_3 = c_4 = 1$, $y_4 = 3$, and $z_4 = 14$. Thus, the coefficients determining the 4-jet of the image cusp agree. Equivalently,

$$\beta_A''(0) = \beta_B''(0), \qquad \beta_A'''(0) = \beta_B'''(0), \qquad \beta_A^{(4)}(0) = \beta_B^{(4)}(0).$$

Hence, the two pair presentations determine the same 4-jet of the distinguished cusp. In particular, they have the same singular normal plane, singular binormal line, singular torsion, and osculating centre.

The adapted-normal-form expression for the fourth-order invariant, however, is derived from different coefficient contributions. For f_A,

$$\Delta_A = 1 - 3 - (6 + 6 + 2) = -16,$$

whereas for f_B,

$$\Delta_B = 1 - 3 - (0 + 12 + 2) = -16.$$

Thus, the value of Δ is the same in both cases, although its decomposition into adapted-normal-form coefficients is different.

This example shows that the fourth-order contact invariant is determined by the 4-jet of the image cusp. The adapted normal form illustrates how the same invariant is realized by a particular presentation of the pair.

6. Conclusions and Further Directions

The results of this study provide a local description of the sphere-contact geometry of cuspidal edge–cusp pairs. Furthermore, they show that the pair viewpoint provides a useful framework for studying contact problems on singular surfaces together with distinguished singular curves.

For cuspidal edge–cusp pairs, the restricted distance squared family provides a natural affine filtration of sphere centres,

$$\Pi_{(X,B)} \supset \ell_{(X,B)} \supset \{q_{\mathrm{osc}}(X, B)\}.$$

The first two levels are determined by the lower-order geometry of the image cusp, while the final level is governed by the fourth-order invariant. When this invariant is nonzero, it determines a unique fourth-order centre. Thus, the singular invariants of the distinguished cusp provide explicit formulas for the normal plane, the binormal line, and the osculating centre.

A central feature of the construction is the separation between the geometric invariant and its adapted-coordinate expression. The fourth-order geometry is determined by the 3- and 4-jets of the image cusp β. In adapted normal form, the same invariant is written in terms of the coefficients of the ambient cuspidal edge and the source curve; thus, the geometry of the cusp is distinguished from its realization in a particular pair presentation.

Several areas for further development are promising, the first of which includes extension of the pair framework beyond $(2, 3)$-cusps. For example, one may consider other singular curve types on a cuspidal edge or formulate analogous contact problems for pairs arising on other wavefront singularities.

A second area to explore is the replacement of the Euclidean distance squared family by other families of contact functions adapted to the pair. Natural candidates include

restricted height, support, and anisotropic distance-type functions. Such variants may lead to corresponding notions of normal, binormal, and osculating data adapted to different ambient geometries.

Furthermore, higher-order contact warrants further investigation. Beyond the A_2-, A_3-, and A_4-strata studied here, a natural subsequent step would be to analyse higher-codimension strata and the versatility of the associated restricted families. This may provide a more complete hierarchy of centres and invariants along the distinguished cusp.

Finally, the sphere-contact construction developed in this study should be closely related to focal sets, offsets, and parallel surfaces along the distinguished cusp. A systematic study of this relation could clarify how the local invariants obtained here appear in the geometry of caustics and parallel fronts and may also be relevant to applications where singular surfaces arise as envelopes or wavefronts.

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