Fierz transformations and renormalization schemes for four-quark operators

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Abstract. It has been shown that the choice of renormalization scheme is crucial for four-quark operators, in particular for neutral kaon mixing beyond the Standard Model. In the context of SMOM schemes, the choice of projector is not unique and is part of the definition of the renormalisation scheme. I present the non-diagonal Fierz relations which relate some of these projectors.

1 Introduction

A significant discrepancy has been observed for the four-quark operators matrix elements needed in the study of neutral kaon oscillations beyond the Standard Model. Beyond the quenched approximation, these quantities have been computed by RBC-UKQCD, [1, 2], by ETM [3, 4] and by SWME [5, 6]. The renormalisation factors have also been studied by Alpha [7]. In collaboration with RBC-UKQCD, we have argued that the renormalisation procedure is responsible for this discrepancy: this was first reported in [8, 9] and published in [10]. We have shown that two SMOM schemes lead to very similar results after conversion to $\overline{\mathrm{MS}}$. However, for two of these quantities, these SMOM results are inconsistent with the ones obtained through the traditional RI-MOM intermediate scheme (with exceptional kinematics). We refer the reader to [11] for a review on the subject at this conference and to [12] for more details on the recent RBC-UKQCD results.

In Fig 1, we show that RI-MOM results are consistent within each other, regardless of the number of dynamical flavours, but significantly below the results obtained though RI-SMOM schemes. The latter are also compatible with SWME [5, 6] where the renormalisation is performed at 1-loop in perturbation theory. In [13], we studied in details the RI-MOM and SMOM schemes, and pointed out potential issues with the RI-MOM procedure with exceptional kinematics. Using non-diagonal Fierz transformation, I show here that the SMOM schemes called (γ_μ, γ_μ) and $(\not{q}, \not{q})$ are mathematically different. These Fierz transformations presented here are general and can be used in different contexts.

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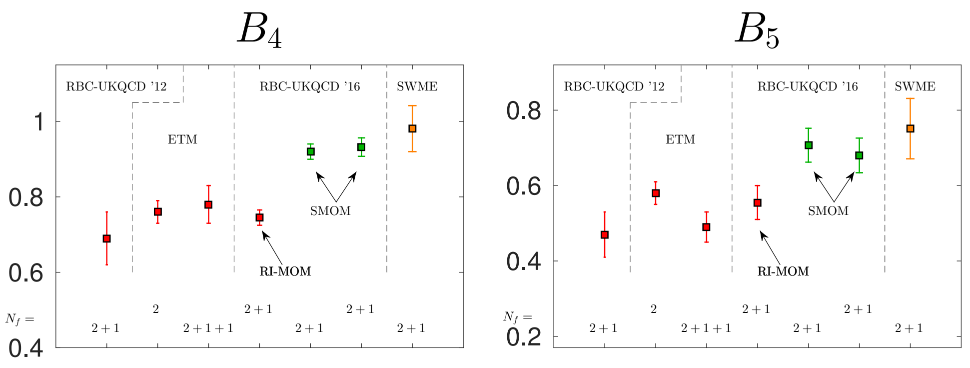

Figure 1. Bag parameters of two BSM matrix elements. We show that the SMOM results (green points) are consistent within each other, but significantly different from the RI-MOM results with exceptional kinematics (red points). The SWME results are obtained through 1-loop perturbative renormalisation and are also compatible with our SMOM results. This effect is more pronounced for B_4 than for B_5 but appears to be systematic. The number of flavours does not seem to play an important role.

2 $\Delta S = 2$ Four quark operators

Within the Standard Model (SM), neutral kaon mixing involves the following *colour unmixed* four-quark operator

$$O_1 = (\bar{s}_a \gamma_\mu (1-\gamma_5) d_a)\,(\bar{s}_b \gamma_\mu (1-\gamma_5) d_b) \equiv (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1-\gamma_5)d)_{(\mathrm{unm})}\,,$$

where a and b are colour indices. The corresponding *colour mixed* operator reads

$$O_1' = (\bar{s}_a \gamma_\mu (1-\gamma_5) d_b)\,(\bar{s}_b \gamma_\mu (1-\gamma_5) d_a) \equiv (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1-\gamma_5)d)_{(\mathrm{mix})}\,. \tag{1}$$

One can show through a Fierz transformation that these operators are identical, ie $O_1' = O_1$.

Beyond the Standard Model, it is customary to introduce four extra operators. Using the same conventions, they can be written as two doublets, for example in the susy basis [14]:

$$O_2 = (\bar{s}(1-\gamma_5)d)\,(\bar{s}(1-\gamma_5)d)_{(\mathrm{unm})}\,, \tag{2}$$
$$O_3 = (\bar{s}(1-\gamma_5)d)\,(\bar{s}(1-\gamma_5)d)_{(\mathrm{mix})}\,, \tag{3}$$

and

$$O_4 = (\bar{s}(1-\gamma_5)d\,(\bar{s}(1+\gamma_5)d)_{(\mathrm{unm})}\,, \tag{4}$$
$$O_5 = (\bar{s}(1-\gamma_5)d\,(\bar{s}(1+\gamma_5)d)_{(\mathrm{mix})}\,. \tag{5}$$

3 Fierz transformation

Denoting by Γ an arbitrary Dirac matrix, the four quark operators have the explicit form

$$(\bar{s}^\alpha_a \Gamma_{\alpha\beta} d^\beta_b)\,(\bar{s}^\gamma_b \Gamma_{\gamma\delta} d^\delta_a) \tag{6}$$

where Greek indices run in Dirac space. Swapping the Dirac indices $\beta \leftrightarrow \delta$ is equivalent to changing the color structure, from colour-mixed to colour-unmixed and vice-versa. However, according to Fierz theorem, one has

$$\Gamma^i_{\alpha\beta}\,\Gamma^j_{\gamma\delta} = -\sum_{k,l} F^{ijkl}\,\Gamma^k_{\alpha\delta}\,\Gamma^l_{\gamma\beta}\,, \tag{7}$$

where the indices i, j, k, l run from 1 to 16, and the 16 matrices Γ^i form a basis of the four-by-four complex matrices. F is therefore a 16^4 tensor, whose entries are a priori unknown.

As a consequence, the colour mixed operators can be expressed in terms of linear combination of colour unmixed operators. We also note that Fierz transformation are properties of the Γ matrices, therefore the Fierz identities are exact on the lattice. However these identities only hold in four-dimension; in $\overline{\mathrm{MS}}$, these identities are modified by the presence of evanescent operators.

3.1 Choice of basis

It is possible to choose a basis in which all the operators are *colour unmixed*

$$Q_2 = (\bar{s}\gamma_\mu(1-\gamma_5)d)\,(\bar{s}\gamma_\mu(1+\gamma_5)d) \tag{8}$$

$$Q_3 = (\bar{s}(1-\gamma_5)d)\,(\bar{s}(1+\gamma_5)d) \tag{9}$$

and

$$Q_4 = (\bar{s}(1-\gamma_5)d)\,(\bar{s}(1-\gamma_5)d) \tag{10}$$

$$Q_5 = \frac{1}{4}(\bar{s}(\sigma_{\mu\nu}(1-\gamma_5))d)\,(\bar{s}(\sigma_{\mu\nu}(1-\gamma_5))d) \tag{11}$$

where we omitted the colour indices. This choice can be found for example in [15].

We now separate the parity sectors, for example $Q_1 = Q_1^+ + Q_1^-$ with

$$Q_1^+ = (\bar{s}\gamma_\mu d)\,(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d)\,(\bar{s}\gamma_\mu\gamma_5 d) = VV + AA \tag{12}$$

using the standard notation

$$\begin{aligned}
SS &= (\bar{s}d)(\bar{s}d)\\
VV &= (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d)\\
TT &= \sum_{\nu>\mu}(\bar{s}\gamma_\mu\gamma_\nu d)(\bar{s}\gamma_\mu\gamma_\nu d)\\
AA &= (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d)\\
PP &= (\bar{s}\gamma_5 d)(\bar{s}\gamma_5 d)
\end{aligned}$$

We can derive the well-known Fierz identities, *in Eucledian space time* we find

$$\begin{pmatrix} VV+AA \\ VV-AA \\ SS-PP \\ SS+PP \\ TT \end{pmatrix}_{(\mathrm{mix})} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 1/2 \\ 0 & 0 & 0 & 3/2 & 1/2 \end{pmatrix} \times \begin{pmatrix} VV+AA \\ VV-AA \\ SS-PP \\ SS+PP \\ TT \end{pmatrix}_{(\mathrm{unm})}, \tag{13}$$

see for example [16]. We note the mixing pattern expected from chiral symmetry is respected, ie $VV + AA$ renormalises multiplicatively, $VV - AA$ and $SS - PP$ mix together, and so do $SS + PP$ and TT. However these identities are a restriction to the *diagonal* case, ie $i = j$ in Eq.(7). Other Fierz relations can also be found in [17].

3.2 Choice of projectors

Following the *Rome-Southampton* method, the renormalisation of a four-quark operators O requires the projection of an amputated green function Λ_O

$$P\left[\Lambda_O\right] = T \ , \tag{14}$$

where T is the tree level value and P is a projector in Dirac and colour space

$$[P]^{ab,cd}_{\alpha\beta} = \left[P^{\text{Dirac}}\right]_{\alpha\beta,\gamma\delta} \left[P^{\text{Colour}}\right]^{ab,cd} \ . \tag{15}$$

In the operator mixing case P and Λ_O are vectors, $P\left[\Lambda_O\right]$ and T are matrices.

As an example, let us consider the standard model operator $VV + AA$. A natural choice for P is

$$\left[P^{\text{Dirac}}\right]_{\alpha\beta,\gamma\delta} = \left[\gamma_\mu\right]_{\alpha\beta} \times \left[\gamma_\mu\right]_{\gamma\delta} + \left[\gamma_\mu\gamma_5\right]_{\alpha\beta} \times \left[\gamma_\mu\gamma_5\right]_{\gamma\delta} \ , \tag{16}$$

$$\left[P^{\text{Colour}}\right]^{ab,cd} = \delta^{ab}\delta^{cd} \equiv P^{\text{unm}} \tag{17}$$

We have defined the projector with the same Dirac-colour structure as the operator. Such a projector is called a γ_μ-projector $\equiv P^{\gamma_\mu}$. Note that we can also define

$$\left[P^{\text{Colour}}\right]^{ab,cd} = \delta^{ad}\delta^{cb} \equiv P^{\text{mix}} \ , \tag{18}$$

However, if a SMOM scheme is implemented, there is a non zero momentum transfer $q = p_2 - p_1$. Therefore we can also define a so-called $\not{q}$-projector [18]

$$P^{\not{q}} = \left[\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5\right] \ P^{\text{Colour}} \ . \tag{19}$$

The choice of Projector is part of the definition of the scheme, they lead to (*a priori*) different non-perturbative Z factors and have different $\overline{\text{MS}}$ conversion factors. (After conversion to $\overline{\text{MS}}$, the Z factors should agree up to truncation error in the perturbative expansion and to discretisation artefacts.).

The "recipe" to define a $\not{q}$-projector for a four-quark operator is the substitution $\gamma_\mu \times \gamma_\mu \to \not{q} \times \not{q}$. For the operators $Q_{3,4} = SS \mp PP$, where no γ_μ structure is present, the trick is to use a Fierz identity [13]

$$P_{SS-PP} \ P^{\text{unm}} = -\frac{1}{2} \ P_{VV-AA} P^{\text{mix}} \ , \tag{20}$$

$$P_{SS+PP} \ P^{\text{unm}} = \frac{1}{2} \left(P_{TT} - P_{SS+PP}\right) \ P^{\text{mix}} \ . \tag{21}$$

For example, for $Q_{2,3}$, we define

$$P_2^{\not{q}} = \frac{2}{q^2}(\not{q}\not{q} - \not{q}_5\not{q}_5) P^{\text{unm}} \tag{22}$$

$$P_3^{\not{q}} = \frac{2}{q^2}(\not{q}\not{q} - \not{q}_5\not{q}_5) P^{\text{mix}} \tag{23}$$

The corresponding Fierz identities for the $\not{q}$-projectors cannot be extracted from Eq.(13) because they involve non-diagonal ($i \neq j$ in Eq.(7)):

$$\not{q} \times \not{q} = q_\alpha q_\beta \left(\gamma_\alpha \times \gamma_\beta\right) \ . \tag{24}$$

4 Non diagonal Fierz identities and $\not{q}$ projectors

We want to know explicitly the relation between the projector of different colour structure

$$P_i^{\not{q}} P^{\rm mix} = F_{ij} P^{\rm unm} \; . \tag{25}$$

For the Standard Model operator, we have [19]

$$(\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5)_{\rm mix} \quad = \quad \left(\frac{q^2}{2} \left(\gamma_\mu \times \gamma_\mu + \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5 \right) - (\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5) \right)_{\rm unm} \; . \tag{26}$$

Therefore

$$\begin{pmatrix} VV + AA \\ \frac{2}{q^2}(\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5) \end{pmatrix}_{\rm mix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} VV + AA \\ \frac{2}{q^2}(\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5) \end{pmatrix}_{\rm unm} \; , \tag{27}$$

which can easily be diagonalised and we find that

$$\frac{2}{q^2} \left(\not{q} \times \not{q} + \not{q}\gamma_5 \times \not{q}\gamma_5\right) - \frac{1}{2} \left(\gamma_\mu \times \gamma_\mu + \gamma_\mu\gamma_5 \times \gamma_\mu\gamma_5 \right) \tag{28}$$

is eigenvector with eigenvalue -1.

We turn now to the case of the $(8,8)$ operators ($Q_{2,3}$ or $O_{4,5}$). We define

$$\not{q}_5 \quad = \quad \not{q}\gamma_5 \; , \tag{29}$$

$$\not{T}\not{T} \quad = \quad q_\mu q_\nu \left(\sigma_{\mu\rho} \times \sigma_{\nu\rho} \right) \; , \tag{30}$$

$$\not{T}_5\not{T}_5 \quad = \quad q_\mu q_\nu \left(\sigma_{\mu\rho}\gamma_5 \times \sigma_{\nu\rho}\gamma_5 \right) \; , \tag{31}$$

and derive the following Fierz matrix

$$\begin{pmatrix} VV - AA \\ SS - PP \\ \frac{2}{q^2}(\not{q}\not{q} - \not{q}_5\not{q}_5) \\ \frac{2}{q^2}(\not{T}\not{T} - \not{T}_5\not{T}_5) \end{pmatrix}_{\rm mix} = \begin{pmatrix} 0 & -2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1/2 \\ 1 & 0 & -2 & 0 \end{pmatrix} \times \begin{pmatrix} VV - AA \\ SS - PP \\ \frac{2}{q^2}(\not{q}\not{q} - \not{q}_5\not{q}_5) \\ \frac{2}{q^2}(\not{T}\not{T} - \not{T}_5\not{T}_5) \end{pmatrix}_{\rm unm} \; . \tag{32}$$

We recognise the γ_μ-projectors in the top-left corner, and we check that the $\not{q}$ are not linear combinations of the γ_μ-projectors. Therefore, the γ_μ and $\not{q}$-projectors defined above (and in [13]) lead to independent renormalisation schemes.

For the $(6,\bar{6})$ operators, the same can be done, but the tensor has be taken with care. Up to parity-odd terms, we have

$$(1-\gamma_5) \times (1-\gamma_5) \quad = \quad 1 \times 1 + \gamma_5 \times \gamma_5 \; , \tag{33}$$

$$\sigma_{\mu\nu}(1-\gamma_5) \times \sigma_{\mu\nu}(1-\gamma_5) \quad = \quad 2\sigma_{\mu\nu} \times \sigma_{\mu\nu} \; , \tag{34}$$

$$q_\mu q_\nu \left(\sigma_{\mu\rho}(1-\gamma_5) \times \sigma_{\nu\rho}(1-\gamma_5) \right) \quad = \quad q_\mu q_\nu \left(\sigma_{\mu\rho} \times \sigma_{\nu\rho} + \sigma_{\mu\rho}\gamma_5 \times \sigma_{\nu\rho}\gamma_5 \right) . \tag{35}$$

The last equation can be written as

$$q_\mu q_\nu \left(\sigma_{\mu\rho} \times \sigma_{\nu\rho} + \sigma_{\mu\rho}\gamma_5 \times \sigma_{\nu\rho}\gamma_5 \right) = \frac{q^2}{2} \sigma_{\mu\nu} \times \sigma_{\mu\nu} \; , \tag{36}$$

which means that the naive definition of a $\not{q}$-projector for the tensor is directly proportional to corresponding γ_μ-projector.

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