Contact of circles with surfaces: answers to a question of Montaldi

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Abstract We answer a question raised by J.Montaldi in this journal in 1986 as to the exact upper bound on the number of circles which can have 5-point contact with a generic smooth surface M in $\mathbb{R}^3$, at a point of M.
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In his seminal study [3] of the contact of circles with smooth surfaces in real euclidean 3-space $\mathbb{R}^3$, James Montaldi observes the following (page 118). *An upper bound on the number of circles which can have at least 5-point contact with a generic surface at any point is 10, but no example exists realizing this maximum.* As part of a complementary study of this topic we have used different approaches to the problem and found such examples; this note explains briefly how to construct them.

Let M be a smooth surface in $\mathbb{R}^3$ given locally in Monge form $z = f(x, y)$ where $f(x, y) = f_{20}x^2 + f_{02}y^2 + \sum_{n\geq 3}\sum_{i=0}^{n} f_{n-i,i}x^{n-i}y^i$; we shall only need $n \leq 4$ in what follows. We shall avoid umbilic points, which are covered separately in [3]; thus $f_{20} \neq f_{02}$. Let Π be the plane $z = ax + by$ (we consider planes through the z-axis separately) and C be the intersection curve $C = \Pi \cap M : f(x, y) = ax + by$; this is locally smooth, parametrized by x or y, for $(a, b) \neq (0, 0)$ (we consider $a = b = 0$, where Π is the tangent plane of M at the origin $\mathbf{O}$, separately). Let $P = (u, v, au + bv)$ be a point of Π. We consider the distance-squared function from P to the curve C, locally to $\mathbf{O}$, and write down the successive conditions that all derivatives of the distance-squared function up to and including the fourth vanish at $\mathbf{O}$. This indicates that C has a 'higher vertex' at $\mathbf{O}$, and that the circle centre P has 5-point contact with M there. Assume $a \neq 0$ and write $\lambda = \frac{b}{a}$; the results are analogous assuming $b \neq 0$. Then the vanishing derivatives up to the *third* allow us to express a, b, u, v as functions of λ and the coefficients f_{ij} up to order 3; the additional zero fourth derivative then results in an equation, containing order 4 coefficients, to be satisfied by λ.

Write A for the expression $f_{03}\lambda^3 - f_{12}\lambda^2 + f_{21}\lambda - f_{30}$ which appears in some denominators below. In fact $A = 0$ is exactly the condition for M intersected with the 'normal plane' $ax + by = 0$ to have a vertex at $\mathbf{O}$, meaning there is a circle with centre on the normal line there and having *four-point* contact with M. See below for further analysis. The other factor $a^2 f_{02} + b^2 f_{20}$ is zero when Π meets the tangent plane $z = 0$ at $\mathbf{O}$ in an *asymptotic line* for M at $\mathbf{O}$. This sends the centre of the circle to infinity: the circle is a straight line.

We find that when C has a vertex at $\mathbf{O}$ (circle with 4-point contact) then

$$a = \frac{2\lambda(f_{02} - f_{20})(f_{20}\lambda^2 + f_{02})}{(\lambda^2 + 1)A},\ b = \lambda a, u = \frac{a(a^2 + b^2)}{2(a^2 + b^2 + 1)(a^2 f_{02} + b^2 f_{20})},\ v = \lambda u.$$

Now the *five point contact* condition is the vanishing of a polynomial $p(\lambda)$ of degree 10, with coefficients polynomials in the f_{ij} up to order four. This polynomial has the following features: (i) there is no term of degree 1 or 9; (ii) $p = f_{30}^2\lambda^{10} + \ldots - f_{03}^2$; (iii) there is one linear relation between the other coefficients, namely the coefficient of λ^5 is the sum of the coefficients of λ^3 and λ^7. We are concerned with the number of real roots of p, seeking values of f_{ij} for which the maximum number 10 is attained.

It is convenient to scale x, y and z in $\mathbb{R}^3$ to make $f_{02} = f_{20} + \frac{1}{2}\sqrt{2}$ (recall $f_{20} \neq f_{02}$). Then p can be re-cast in the form

$$q(\lambda) = \lambda^8 + (\lambda^2+1)(a_8^2\lambda^8 + a_6\lambda^6 + a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 - a_0^2),$$

where the coefficients a_i can be expressed in terms of the f_{ij}. For example, $a_8 = f_{30}$ and $a_0 = f_{03}$. Conversely, the f_{ij} up to order 4 can be expressed in terms of the a_i, with f_{20}, f_{21} and f_{12} arbitrary and the rest determined by these choices. (Note that because of this we can always avoid $A = 0$ for any of the solutions λ.) As an example.

$$f_{13} = \tfrac{1}{4}(-4a_0a_8 + 8a_0f_{12} - 4f_{12}f_{21} - a_3)\sqrt{2}.$$

All this means that we are now looking for real roots λ of q. A solution is found by means of the Direct Search algorithm created by Sergey Moiseev using the program Maple [2]. This has produced many examples with 10 real roots, including the following one:

$$a_0 = 0.079, a_2 = 0.2361, a_3 = -0.8598, a_4 = 1.0035, a_5 = 0.1733, a_6 = -1.0484, a_8 = 0.0298$$

which gives the following for f:

$$x^2+(1+\tfrac{1}{2}\sqrt{2})y^2-0.0298x^3+3x^2y+2xy^2-0.079y^3+4.12115x^4+8.6802x^3y-1.5571x^2y^2-8.6315xy^3+3.82275y^4,$$

choosing $f_{20} = 1, f_{12} = 2, f_{21} = 3$. This shows that indeed *surfaces exist for which there are 10 circles at a point each of which has 5-point contact with the surface there.* And of course the coefficients can be perturbed slightly without affecting this outcome.

It remains to ask whether the number 10 can be increased by considering (i) sections of M by planes $ax+by = 0$ ('normal planes') and (ii) circles lying in the tangent plane $z = 0$. (For details of (ii), see [1].) In fact we find that in each case any 5-point contact circle occurring in these situations automatically subtracts from the 10 occurring for general planes: for example in case (ii) one solution of the degree 10 equation for λ satisfies $f_{20}\lambda^2 + f_{02} = 0$ so that a and b are both 0, that is the plane $z = ax + by$ of the circle is actually the tangent plane.

Therefore 10 is an upper bound on the number of 5-point contact circles and this can be achieved by a generic surface M.

References

[1] Peter Giblin, Graham Reeve and Ricardo Uribe-Vargas, 'Contact with circles and Euclidean invariants of smooth surfaces in $\mathbb{R}^3$', *Quarterly J. Math. (Oxford)* (published 20 March 2022) https://doi.org/10.1093/qmath/haab058.

[2] S.N.Moiseev, 'Universal derivative-free optimization method with quadratic convergence', arXiv:1102.1347v1. [math.OC]. Feb 7, 2011.

[3] James A. Montaldi, 'Surfaces in 3-space and their contact with circles', *J. Diff. Geom.* 23 (1986), 109–126.

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