Parallel Contextual Array Insertion Deletion P Systems and Siromoney Matrix Grammars

S. Jayasankar [a], D. Gnanaraj Thomas[b], S. James Immanuel[b], Meenakshi Paramasivan[c], T. Robinson[d] and Atulya K. Nagar[e]

[a]Department of Mathematics, Ramakrishna Mission Vivekananda College, Mylapore, Chennai - 600004, India [b]Department of Science and Humanities (Mathematics Division), Saveetha School of Engineering, Chennai - 602105, India [c]Department of Computer Science, University of Trier, Germany [d]Department of Mathematics, Madras Christian College, Tambaram, Chennai - 600059, India [e]Department of Mathematics and Computer Science, Liverpool Hope University, Liverpool, United Kingdom

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ABSTRACT
James et al. have introduced a variant of P systems called parallel contextual array insertion deletion P system and studied some of its properties. The family of array languages generated by this variant of $PCAIDPS$ includes families of array languages like recognizable picture languages (REC) and context-sensitive matrix languages $\mathfrak{L}(CSML)$. In this paper we show that another interesting family $\mathfrak{L}(CF : RIR)$ of Siromoney matrix languages is included in the family of array languages generated by $PCAIDPS$ with two membranes.

KEYWORDS
P System; Rectangular array; Contextual array insertion and deletion; Siromoney matrix grammar.

1. Introduction

A P system or a membrane system introduced by Gh. Paun [21] evolves in parallel. A computation starts from an initial configuration of a system, defined by a membrane structure with objects and evolution rules in each membrane and terminates when no further rule can be applied. Various types of P systems have been introduced in the literature and their properties, computing power, normal forms and basic decision problems have been studied [22].

The study of P systems (Membrane Computing (MC)) can be broadly classified into at least three categories:

(i) Theoretical aspects of MC.
(ii) Applications of MC.
(iii) Implementation of MC.

Recently, Petr Sosik [23] has given a suvey on models of P systems which solve hard

CONTACT S. Jayasankar. Email: ksjayjay@gmail.com

problems belonging to the classes containing NP. Rudolf Freund in [24] has explained how derivation models and halting conditions may influence the computational power of P systems. It has been shown that P systems with proteins on membranes and separation rules are efficient to obtain solutions to QBF-SAT [5]. James Cooper [13] provided an alternative compact single-cell solutions to the graph 3-colouring problem using P systems with compound objects. Ignacio Pérez-Hurtado et al [12] has presented a compiler for membrane computing. The input P systems are written in a common language called P-Lingua.

There are various applications of P systems in different fields have been summarised in [9]. To mention a few, we see P systems

(i) modelling swarming and aggregating in a Myxobacterial colony [1].
(ii) relating with P colonies [19].
(iii) designing robot controllers [2].
(iv) relating to biological phenomenon that neurons communicating via spikes [29].

To take a turn to the field of image analysis, Daniel Díaz-Pernil et al [4] surveyed applications of P systems in image processing. In this paper, we deal with applications of P systems in terms of rectangular arrays of letters yielding pictures or images by replacing letters by primitive symbols. Array rewriting P systems were introduced by R. Ceterchi et al. [3] by combining the branches of membrane computing and picture grammars. In [6], a P system model called contextual array P system with array objects and array contextual rules has been introduced similar to the contextual way of handling string objects in P systems [20]. An interesting model of array P systems using contextual array grammars [8] with an application of Kolam patterns has been considered in [7]. In [14], another P system model namely, external and internal parallel contextual array P systems have been defined and examined. A new variant of P system model called parallel contextual array insertion deletion P system (PCAIDPS) has been studied in [15] based on array insertion and deletion operations and parallel contextual array insertion deletion grammar [16]. It is proved that the family of array languages generated by PCAIDPS includes families of array languages like recognizable picture languages (REC) [10,11] and context-sensitive matrix languages $\mathfrak{L}(CSMG)$ [25]. For an application of insertion and deletion operations in natural computing, we refer to [17] and in P systems, we refer to [18].

Insertion Deletion System has the generative power equal to the class of recursive enumerable string languages [28]. The primary motivation for studying PCAIDPS is not only to develop the 2D counterpart of Insertion Deletion System available for one dimensional string languages but also to find the status of $\mathfrak{L}(PCAIDPS)$ among the families of 2D languages available in the literature. As far as our knowledge goes, there is no definite hierarchy for the families of 2D languages like Chomsky hierarchy for the classes of string languages. Hence, we are motivated to show that PCAIDPS has more generative power than any other class of 2D grammars generating languages having intersection with REC and $\mathfrak{L}(CSMG)$. One such family is $\mathfrak{L}(CF : RIR)$ of Subramanian et al [27] who have considered the Siromoney matrix grammar (SMG) [25] and examined the notion of attaching indices to nonterminals in the vertical derivations. The system (CF:RIR)SMG has greater generative power compared to context free matrix grammars (CFMG) [25] with interesting applications in the studies of tilings, polyominoes, noisy patterns and parquet deformations. In this paper we prove that the family $\mathfrak{L}(PCAIDPS_2)$ includes the family $\mathfrak{L}(CF : RIR)$ [27].

2. Preliminaries

In this section, we recall some notions of parallel contextual array insertion deletion P systems and give an example. For further details of the P system we can refer to [15].

Let V be a finite alphabet, V^*, the set of words over V including the empty word λ. $V^+ = V^* - \{\lambda\}$. For $w \in V^*$ and $a \in V$, $|w|_a$ denotes the number of occurrences of a in w. An array consists of finitely many symbols from V that are arranged as rows and columns in some particular order and is written in the form, $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ or $\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{matrix}$ or in short $A = [a_{ij}]_{m \times n}$, for all $a_{ij} \in V$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. The set of all arrays over V is denoted by V^{**} which also includes the empty array Λ (zero rows and zero columns). $V^{++} = V^{**} - \{\Lambda\}$. The column concatenation of $A = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nq} \end{bmatrix}$, defined only when $m = n$, is given by $A \oplus B = \begin{bmatrix} a_{11} & \cdots & a_{1p} & b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} & b_{n1} & \cdots & b_{nq} \end{bmatrix}$. As $1 \times n$-dimensional arrays can be easily interpreted as words of length n (and vice versa), we will then write their column concatenation by juxtaposition (as usual). Similarly, the row concatenation of A and B, defined only when $p = q$, is given by $A \ominus B = \begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mp} \\ b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nq} \end{bmatrix}$. The empty array acts as the identity for column and row concatenation of arrays of arbitrary dimensions.

Definition 2.1. Let V be a finite alphabet. A column array context over V is of the form, $c = \left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in V^{**}$, u_1, u_2 are of size $1 \times p$, $p \geq 1$.

A row array context over V is of the form, $r = \left[\begin{smallmatrix} u_1 & u_2 \end{smallmatrix}\right] \in V^{**}$, u_1, u_2 are of size $p \times 1$, $p \geq 1$.

Definition 2.2. The parallel column contextual insertion (deletion) operation is defined as follows: Let V be an alphabet, C be a finite subset of V^{**} whose elements are the column array contexts and $\varphi_c^i(\varphi_c^d) : V^{**} \times V^{**} \to 2^C$ be a choice mapping. We define $\varphi_c^i(\varphi_c^d) : V^{**} \times V^{**} \to 2^{V^{**}}$ such that, for arrays $A = \left[\begin{smallmatrix} a_{1j} & \cdots & a_{1(k-1)} \\ \vdots & \ddots & \vdots \\ a_{mj} & \cdots & a_{m(k-1)} \end{smallmatrix}\right]$,

$B = \left[\begin{smallmatrix} a_{1k} & \cdots & a_{1(l-1)} \\ \vdots & \ddots & \vdots \\ a_{mk} & \cdots & a_{m(l-1)} \end{smallmatrix}\right] \left(B = \left[\begin{smallmatrix} a_{1(k-p)} & \cdots & a_{1(l-1)} \\ \vdots & \ddots & \vdots \\ a_{m(k-p)} & \cdots & a_{m(l-1)} \end{smallmatrix}\right]\right), j < k < l, a_{ij} \in V,$

$I_c \in \varphi_c^i(A, B)(\varphi_c^d(A, B)), I_c(D_c) = \left[\begin{smallmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{smallmatrix}\right]$, if $c_i = \left[\begin{smallmatrix} u_i \\ u_{i+1} \end{smallmatrix}\right] \in$

$\varphi_c^i\left(\begin{smallmatrix} a_{ij} & \cdots & a_{i(k-1)} \\ a_{(i+1)j} & \cdots & a_{(i+1)(k-1)} \end{smallmatrix}, \begin{smallmatrix} a_{ik} & \cdots & a_{i(l-1)} \\ a_{(i+1)k} & \cdots & a_{(i+1)(l-1)} \end{smallmatrix}\right)\left(\varphi_c^d\left(\begin{smallmatrix} a_{ij} & \cdots & a_{i(k-1)} \\ a_{(i+1)j} & \cdots & a_{(i+1)(k-1)} \end{smallmatrix}, \begin{smallmatrix} a_{i(k+p)} & \cdots & a_{i(l-1)} \\ a_{(i+1)(k+p)} & \cdots & a_{(i+1)(l-1)} \end{smallmatrix}\right)\right)$, $c_i \in C, 1 \leq i \leq m-1$, not all need to be distinct.

Given an array $X = [a_{ij}]_{m \times n}, a_{ij} \in V$, $X = X_1 \circledcirc A \circledcirc B \circledcirc X_2$ $(X = X_1 \circledcirc A \circledcirc D_c \circledcirc B \circledcirc X_2)$,
$X_1 = \begin{bmatrix} a_{11} & \cdots & a_{1(j-1)} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(j-1)} \end{bmatrix}, A = \begin{bmatrix} a_{1j} & \cdots & a_{1(k-1)} \\ \vdots & \ddots & \vdots \\ a_{mj} & \cdots & a_{m(k-1)} \end{bmatrix}, B = \begin{bmatrix} a_{1k} & \cdots & a_{1(l-1)} \\ \vdots & \ddots & \vdots \\ a_{mk} & \cdots & a_{m(l-1)} \end{bmatrix}, X_2 = \begin{bmatrix} a_{1l} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{ml} & \cdots & a_{mn} \end{bmatrix}$,
$1 \leq j \leq k < l \leq n+1$ (or) $1 \leq j < k \leq l \leq n+1$, we write $X \Rightarrow^{col_i(col_d)} Y$ if $Y = X_1 \circledcirc A \circledcirc I_c \circledcirc B \circledcirc X_2$ $(Y = X_1 \circledcirc A \circledcirc B \circledcirc X_2)$, such that $I_c \in \varphi_c^i(A, B)$ $(D_c \in \varphi_c^d(A, B))$. $I_c(D_c)$ is called as the inserted (deleted) column context. We say that Y is obtained from X by parallel column contextual insertion (deletion) operation. The following 4 special cases for $X = X_1 \circledcirc A \circledcirc B \circledcirc X_2$ are also considered,

(1) For $j = 1$ we have $X_1 = \Lambda$.
(2) For $j = k$, we have $A = \Lambda$. If $j = k = 1$, then $X_1 = \Lambda$ and $A = \Lambda$.
(3) For $k = l$ $(For\ k + p = l)$, we have $B = \Lambda$.
(4) For $l = n + 1$, we have $X_2 = \Lambda$. If $k = l = n + 1$ (If $(k + p) = l = n + 1$), then $B = \Lambda$ and $X_2 = \Lambda$.

The case $j = k = l$ is not considered for parallel column contextual insertion (deletion) operation.

Similarly, we can define parallel row contextual insertion (deletion) operation by inserting (deleting) row context $\mathbf{I_r}(\mathbf{D_r})$ in between two sub-arrays A and B with the help of row operation $\ominus$ and set of row array contexts R. We have $X \Rightarrow^{row_i(row_d)} Y$ if $X = X_1 \ominus A \ominus B \ominus X_2$ $(X_1 \ominus A \ominus D_r \ominus B \ominus X_2)$ and $Y = X_1 \ominus A \ominus I_r \ominus B \ominus X_2$ $(X_1 \ominus A \ominus B \ominus X_2)$.

Definition 2.3. A parallel contextual array insertion deletion P system with h membranes $(PCAIDPS_h)$ is a construct,

$$\prod = (V, T, \mu, C, R, (M_1, I_1, D_1), \ldots, (M_h, I_h, D_h), \varphi_c^i, \varphi_r^i, \varphi_c^d, \varphi_r^d, i_0)$$

where,

– V is the finite nonempty set of symbols called alphabet;
– $T \subseteq V$ is the output alphabet;
– μ is the membrane structure with h membranes or regions;
– C is the finite subset of V^{**} called set of column array contexts;
– R is the finite subset of V^{**} called set of row array contexts;
– M_i is the finite set of arrays over V called as axioms associated with the region $\mu_\mathbf{i}$ of μ;
– $\varphi_c^i : V^{**} \times V^{**} \to 2^C$ is the choice mapping performing parallel column contextual insertion operations;
– $\varphi_r^i : V^{**} \times V^{**} \to 2^R$ is the choice mapping performing parallel row contextual insertion operations;
– $\varphi_c^d : V^{**} \times V^{**} \to 2^C$ is the choice mapping performing parallel column contextual deletion operations;
– $\varphi_r^d : V^{**} \times V^{**} \to 2^R$ is the choice mapping performing parallel row contextual deletion operations;
– $I_i = \emptyset$ (or) $\left\{ \left(\left\{ \varphi_c^i(A_i, B_i) = \left[\begin{smallmatrix} u_i \\ u_{i+1} \end{smallmatrix}\right] \middle| i = 1, 2, \ldots, m-1 \right\}, \alpha \right) \right\}$ where
$A_i = \left[\begin{smallmatrix} a_{ij} & \cdots & a_{i(k-1)} \\ a_{(i+1)j} & \cdots & a_{(i+1)(k-1)} \end{smallmatrix}\right], B_i = \left[\begin{smallmatrix} a_{ik} & \cdots & a_{i(l-1)} \\ a_{(i+1)k} & \cdots & a_{(i+1)(l-1)} \end{smallmatrix}\right]$, $1 \leq j \leq k < l \leq n+1$ (or)

$1 \leq j < k \leq l \leq n+1$, $\alpha \in \{here, out, in_t\}$, u_i and u_{i+1} are of size $1 \times p$ with $p \geq 1$.

(or)

$\left\{\left(\left\{\varphi_r^i(C_i, E_i) = \left[\begin{smallmatrix} u_i & u_{i+1} \end{smallmatrix}\right] \middle| i = 1, 2, \ldots, n-1\right\}, \alpha\right)\right\}$ where

$C_i = \left[\begin{smallmatrix} a_{ji} & a_{j(i+1)} \\ \vdots & \vdots \\ a_{(k-1)i} & a_{(k-1)(i+1)} \end{smallmatrix}\right], E_i = \left[\begin{smallmatrix} a_{ki} & a_{k(i+1)} \\ \vdots & \vdots \\ a_{(l-1)i} & a_{(l-1)(i+1)} \end{smallmatrix}\right]$, $1 \leq j \leq k < l \leq m+1$ (or) $1 \leq j < k \leq l \leq m+1$, $\alpha \in \{here, out, in_t\}$, u_i and u_{i+1} are of size $p \times 1$ with $p \geq 1$.

$D_i = \emptyset$ (or) $\left\{ \left(\left\{\varphi_c^d(A_i, B_i) = \left[\begin{smallmatrix} u_i \\ u_{i+1} \end{smallmatrix}\right] \middle| i = 1, 2, \ldots, m-1\right\}, \alpha\right) \right\}$ where

$A_i = \left[\begin{smallmatrix} a_{ij} & \cdots & a_{i(k-1)} \\ a_{(i+1)j} & \cdots & a_{(i+1)(k-1)} \end{smallmatrix}\right], B_i = \left[\begin{smallmatrix} a_{i(k+p)} & \cdots & a_{i(l-1)} \\ a_{(i+1)(k+p)} & \cdots & a_{(i+1)(l-1)} \end{smallmatrix}\right]$, $1 \leq j \leq k < l \leq n+1$, $\alpha \in \{here, out, in_t\}$, u_i and u_{i+1} are of size $1 \times p$ with $p \geq 1$.

(or)

$\left\{\left(\left\{\varphi_r^d(C_i, E_i) = \left[\begin{smallmatrix} u_i & u_{i+1} \end{smallmatrix}\right] \middle| i = 1, 2, \ldots, n-1\right\}, \alpha\right)\right\}$ where

$C_i = \left[\begin{smallmatrix} a_{ji} & a_{j(i+1)} \\ \vdots & \vdots \\ a_{(k-1)i} & a_{(k-1)(i+1)} \end{smallmatrix}\right], E_i = \left[\begin{smallmatrix} a_{(k+p)i} & a_{(k+p)(i+1)} \\ \vdots & \vdots \\ a_{(l-1)i} & a_{(l-1)(i+1)} \end{smallmatrix}\right]$, $1 \leq j \leq k < l \leq m+1$, $\alpha \in \{here, out, in_t\}$, u_i and u_{i+1} are of size $p \times 1$ with $p \geq 1$.

− i_0 is the output membrane.

The array language generated by $\prod$ is denoted by $L(\prod)$ and the family of array languages generated by $PCAIDPS$ with h membranes is denoted by $\mathfrak{L}(PCAIDPS_h)$.

Example 2.4. Consider a P system given by $PCAIDPS_2$

$\prod = (\mathbf{V}, \mathbf{T}, \mu, \mathbf{C}, \mathbf{R}, (\mathbf{M_1}, \mathbf{I_1}, \mathbf{D_1}), (\mathbf{M_2}, \mathbf{I_2}, \mathbf{D_2}), \varphi_\mathbf{c}^\mathbf{i}, \varphi_\mathbf{r}^\mathbf{i}, \varphi_\mathbf{c}^\mathbf{d}, \varphi_\mathbf{r}^\mathbf{d}, \mathbf{1})$,where

$\mathbf{V} = \{\bullet, X, Y\}$; $\mathbf{T} = \{\bullet, X\}$; $\mu = [_1[_2]_2]_1$;

$$\mathbf{C} = \left\{ \begin{bmatrix} \bullet \\ X \end{bmatrix}, \begin{bmatrix} X \\ \bullet \end{bmatrix}, \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}, \begin{bmatrix} Y \\ Y \end{bmatrix} \right\};$$

$$\mathbf{R} = \left\{ \begin{bmatrix} X & Y \\ Y & Y \end{bmatrix}, \begin{bmatrix} Y & \bullet \\ Y & Y \end{bmatrix}, \begin{bmatrix} \bullet & X \\ Y & Y \end{bmatrix}, \begin{bmatrix} \bullet & \bullet \\ Y & Y \end{bmatrix}, \begin{bmatrix} Y & Y \\ X & Y \end{bmatrix}, \begin{bmatrix} Y & Y \\ Y & \bullet \end{bmatrix}, \begin{bmatrix} Y & Y \\ \bullet & X \end{bmatrix}, \begin{bmatrix} Y & Y \\ \bullet & \bullet \end{bmatrix}, \begin{bmatrix} Y & Y \end{bmatrix} \right\};$$

$\mathbf{M_1} = \emptyset$; $\mathbf{I_1} = \emptyset$; $\mathbf{D_1} = \emptyset$;

$$\mathbf{M_2} = \left\{ \begin{bmatrix} X & \bullet & X \\ X & X & X \\ X & \bullet & X \end{bmatrix} \right\};$$

We now define the insertion rules of $\mathbf{I_2}$ as follows:

$$\text{(IR1)} \left\{ \left\{ \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & \bullet \\ X & , & X \end{bmatrix} = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}, \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & \bullet \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix}, \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & X \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} \lambda & \lambda \\ \lambda & \lambda \end{bmatrix} \right\}, out \right\},$$

$$\text{(IR2)} \left\{ \left\{ \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & \bullet \\ X & , & X \end{bmatrix} = \begin{bmatrix} \bullet \\ X \end{bmatrix}, \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & \bullet \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}, \varphi_\mathbf{c}^\mathbf{i} \begin{bmatrix} X & & X \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} X \\ \bullet \end{bmatrix} \right\}, here \right\},$$

(IR3) $\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} X & & \bullet \\ X & , & X \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} X & & \bullet \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \right.\right.$

$\left.\left.\varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} X & & X \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}\right\}, here\right\},$

(IR4) $\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} X & Y & , & X & Y \end{bmatrix} = \begin{bmatrix} X & Y \\ Y & Y \end{bmatrix}, \varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} Y & \bullet & , Y & X \end{bmatrix} = \begin{bmatrix} Y & \bullet \\ Y & Y \end{bmatrix}, \right.\right.$

$\left.\left.\varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} \bullet & X & , & X & X \end{bmatrix} = \begin{bmatrix} \bullet & X \\ Y & Y \end{bmatrix}, \varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} \bullet & \bullet & , & X & X \end{bmatrix} = \begin{bmatrix} \bullet & \bullet \\ Y & Y \end{bmatrix}\right\}, here\right\},$

(IR5) $\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} X & Y & , & X & Y \end{bmatrix} = \begin{bmatrix} Y & Y \\ X & Y \end{bmatrix}, \varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} Y & X & , Y & \bullet \end{bmatrix} = \begin{bmatrix} Y & Y \\ Y & \bullet \end{bmatrix}, \right.\right.$

$\left.\left.\varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} X & X & , & \bullet & X \end{bmatrix} = \begin{bmatrix} Y & Y \\ \bullet & X \end{bmatrix}, \varphi^{\mathbf{i}}_{\mathbf{r}}\begin{bmatrix} X & X & , & \bullet & \bullet \end{bmatrix} = \begin{bmatrix} Y & Y \\ \bullet & \bullet \end{bmatrix}\right\}, here\right\};$

We now define deletion rules of $\mathbf{D_2}$ as follows:

(DR1) $\left\{\left\{\varphi^{\mathbf{d}}_{\mathbf{c}}\begin{bmatrix} X & & \bullet \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \varphi^{\mathbf{d}}_{\mathbf{c}}\begin{bmatrix} X & & \bullet \\ Y & , & Y \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \varphi^{\mathbf{d}}_{\mathbf{c}}\begin{bmatrix} Y & & Y \\ X & , & X \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \right.\right.$

$\left.\left.\varphi^{\mathbf{d}}_{\mathbf{c}}\begin{bmatrix} X & & X \\ Y & , & Y \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}, \varphi^{\mathbf{d}}_{\mathbf{c}}\begin{bmatrix} Y & & Y \\ X & , & \bullet \end{bmatrix} = \begin{bmatrix} Y \\ Y \end{bmatrix}\right\}, here\right\},$

(DR2) $\left\{\left\{\varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} X & \bullet & , & X & X \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}, \varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} \bullet & \bullet & , X & X \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}, \right.\right.$

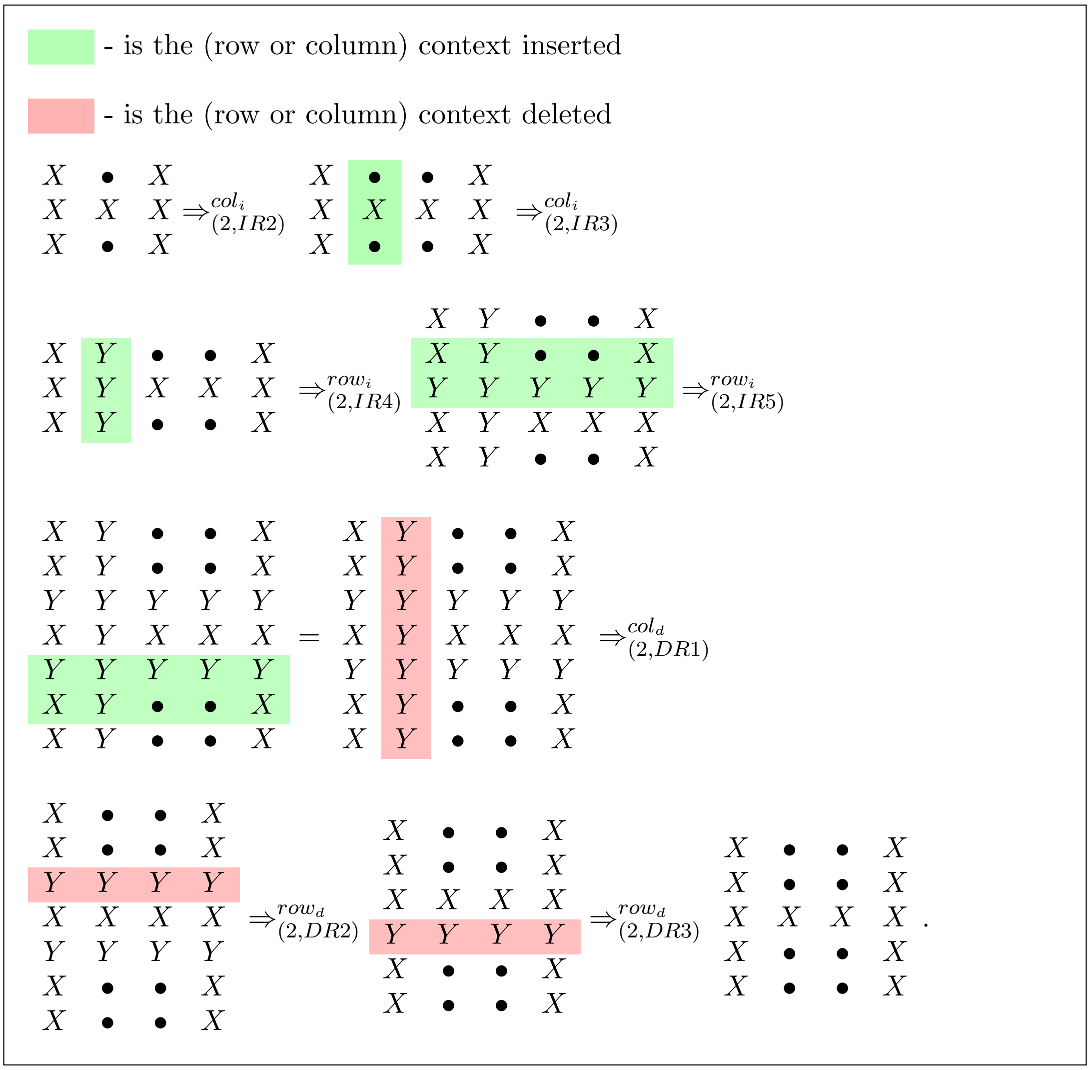
$\left.\left.\varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} \bullet & X & , & X & X \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}\right\}, here\right\},$

(DR3) $\left\{\left\{\varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} X & X & , & X & \bullet \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}, \varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} X & X & , \bullet & \bullet \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}, \right.\right.$

$\left.\left.\varphi^{\mathbf{d}}_{\mathbf{r}}\begin{bmatrix} X & X & , & \bullet & X \end{bmatrix} = \begin{bmatrix} Y & Y \end{bmatrix}\right\}, here\right\};$

Clearly, $L(\prod) = \left\{ \begin{matrix} X & \bullet & X \\ X & X & X \\ X & \bullet & X \end{matrix}, \begin{matrix} X & \bullet & \bullet & X \\ X & \bullet & \bullet & X \\ X & X & X & X \\ X & \bullet & \bullet & X \\ X & \bullet & \bullet & X \end{matrix}, \cdots \right\}$, the set of tokens of H of $X's$ with the horizontal row of $X's$ exactly in the middle. A sample computation is shown below.

In the sample derivation given below, $A \Rightarrow^{\alpha_k}_{(s,r)} B$ means A derives B using (k, α) rewriting rule where k is either insertion (i) or deletion (d), α is either row (row) or column (col) in the membrane s depending upon the contexts and r can be either a set of insertion rules (IRi), $i \in \{1, 2, 3, 4, 5\}$ or a set of deletion rules (DRj), $j \in \{1, 2, 3\}$. Computations in M_2 are performed as follows:



3. Main Result

In this section, we recall the definition of the Siromoney matrix grammar $(CF : RIR)$SMG and an example [27]. We show that $\mathfrak{L}(CF : RIR) \subseteq \mathfrak{L}(PCAIDPS_2)$.

Definition 3.1. A right-linear indexed right-linear(RIR) grammar G is a five-tuple (N,T,F,P,S) where N, T, F are finite, pairwise disjoint sets of nonterminals, terminals and indices respectively; P consists of two types of productions, namely

(1) right-linear productions of the form $A \to xBf$, $A \in N$, $x \in T \cup \{\lambda\}$, $f \in F \cup \{\lambda\}$, $B \in N \cup \{\lambda\}$, not all λ, and
(2) indexed productions of the form $Af \to xB$, $A \in N$, $B \in N \cup \{\lambda\}$, $x \in T \cup \{\lambda\}$, $f \in F$, and $S \in N$ is the start symbol.

Derivations are as in a Chomskian right-linear grammar except that a nonterminal A is replaced using a production of the form (1) or an index f is consumed by a production of the form (2).

Definition 3.2. A (CF:RIR)SMG $G = (G_1, G_2)$ is defined as follows: $G_1 = (N, I, P, S)$ is a context-free grammar (CF) where N is a finite set of non-terminals, I is a finite set of k intermediate symbols $S_1, S_2, \cdots, S_k$ with $N \cap I = \emptyset$, P is a finite set of production rules and $S \in N$ is the start symbol of G_1. G_1 is called the horizontal grammar.

$G_2 = \{G_{21}, G_{22}, G_{23}, \cdots, G_{2k}\}$, where each $G_{2i}, i = 1, \cdots, k$ is called a vertical grammar where $G_{2i} = (N_i, T, F_i, P_i, S_i)$ is a RIR grammar with N_i, F_i being distinct from N_j, F_j for $i \neq j (1 \leq i, j \leq k)$. The intermediate symbol S_i of G_1 is in N_i and is the start symbol of G_{2i}.

The derivations are as follows: G_1 generates finite strings of intermediates as in a Chomskian CF grammar. The vertical derivation starts with a string of intermediates generated by G_2 and proceeds in parallel as in Siromoney matrix grammar [25]. Rules of the same type - either CF productions $A \to \alpha$ or indexed productions $Af \to \alpha$ with the length of α being the same $(3, 2, 1$ or $0)$, are applied in the vertical derivation. At every step of the vertical derivation, a row $A_1 A_2 \cdots A_m$ of non-terminals is expanded or a string of indexed non-terminals $\begin{matrix} A_1 & \cdots & A_m \\ f_1 & \cdots & f_m \end{matrix}$ is rewritten, consuming the indices. A derivation successfully ends on generating an array M over T by rewriting a row of non-terminals with(without) indices by rules of the form $Af \to \alpha$ $(A \to \alpha)$. The array language generated by the grammar G is given by

$$L(G) = \{M \in T^{++} / S \Rightarrow^*_{G_1} \alpha \text{ and } \alpha \Rightarrow^*_{G_2} M, \alpha \in I^{++}\}.$$

Let $\mathfrak{L}(X : RIR)$ be the family of array languages generated by the Siromoney matrix grammars $(X : RIR)$, where $X \in \{R, CF\}$. Here R stands for Regular grammar.

Example 3.3. The $(R : RIR)SMG$ which generates the tokens H of $x's$ with the horizontal row of $x's$ exactly in the middle is as follows:

$$G = (G_1, G_2)$$

where

$$G_1 = (\{S, A\}, \{S_1, S_2\}, \{S \to S_1 A, A \to S_2 A, A \to S_2 S_1\}, S)$$

generates strings of intermediates $S_1 S_2^n S_1$ for $n \geq 1$ and $G_2 = \langle G_{21}, G_{22} \rangle$
where

$$G_{21} = (\{S_1, A_1, A_2, A_3\}, \{x\}, \{g_1, g_2\}, P_{21}, S_1)$$

with $P_{21} = \{S_1 \to x A_1 g_2, A_1 \to A_2 g_1, A_1 \to x A_1 g_1, A_2 g_1 \to x A_3,$
$A_3 g_1 \to x A_3, A_3 g_2 \to x\},$

$$G_{22} = (\{S_2, B_1, B_2, B_3, B_4\}, \{\cdot, x\}, \{f_1, f_2\}, P_{22}, S_2)$$

with $P_{22} = \{S_2 \to \cdot B_1 f_2, B_1 \to \cdot B_1 f_1, S_2 \to \cdot B_2 f_2, B_1 \to \cdot B_2 f_1,$
$B_2 \to B_3 f_1, B_3 f_1 \to x B_4, B_4 f_1 \to \cdot B_4, B_4 f_2 \to \cdot\}.$

A sample vertical derivation is as follows:

$$S_1 S_2 S_1 \Rightarrow_{G_2} \begin{matrix} x & \cdot & x \\ A_1 & B_2 & A_1 \\ g_2 & f_2 & g_2 \end{matrix} \Rightarrow_{G_2} \begin{matrix} x & \cdot & x \\ A_2 & B_3 & A_2 \\ g_1 & f_1 & g_1 \\ g_2 & f_2 & g_2 \end{matrix} \Rightarrow_{G_2} \begin{matrix} x & \cdot & x \\ x & x & x \\ A_3 & B_4 & A_3 \\ g_2 & f_2 & g_2 \end{matrix} \Rightarrow_{G_2} \begin{matrix} x & \cdot & x \\ x & x & x \\ x & \cdot & x \end{matrix}$$

Example 2.4 and example 3.3 exhibit the fact that $\mathfrak{L}(R:RIR) \cap \mathfrak{L}(PCAIDPS_2) \neq \emptyset$. We now proceed to prove the main result.

Theorem 3.4. $\mathfrak{L}(CF:RIR) \subset \mathfrak{L}(PCAIDPS_2)$

Proof. This result can be proved by showing that for any $(CF:RIR)SMG$ grammar G generating a language L, a $PCAIDPS_2$ $\prod$ can be constructed to generate L by introducing insertion deletion rules corresponding to the rules of the grammar G.

For every $(CF:RIR)SMG$, $G = (G_1, G_2)$ with $G_1 = (N, I, P, S)$, $G_2 = \langle G_{21}, \cdots, G_{2k}\rangle$, $G_{2i} = (N_i, T, \mathcal{F}_i, P_i, S_i), N_i, \mathcal{F}_i$ being distinct from $N_j, \mathcal{F}_i$ with $i \neq j$, $i = 1, 2, \cdots, k$ $(1 \leq i, j \leq k)$, we can construct a $PCAIDPS_2$

$$\prod = (\mathbf{V}, \mathbf{T}, [_1[_2]_2]_1, \mathbf{C}, \mathbf{R}, (\mathbf{M_1}, \mathbf{I_1}, \mathbf{D_1}), (\mathbf{M_2}, \mathbf{I_2}, \mathbf{D_2}), \varphi^{\mathbf{i}}_{\mathbf{c}}, \varphi^{\mathbf{i}}_{\mathbf{r}}, \varphi^{\mathbf{d}}_{\mathbf{c}}, \varphi^{\mathbf{d}}_{\mathbf{r}}, \mathbf{1})$$

such that $L(\prod) = L(G)$ where $\mathbf{V} = N \cup (\bigcup_{i=1}^{k} N_i) \cup T \cup (\bigcup_{i=1}^{k} F_i) \cup I \cup \{\#\}$;
$\mathbf{T} = T$; $\mathbf{M_1} = \emptyset$; $\mathbf{I_1} = \emptyset$, $\mathbf{D_1} = \emptyset$; $\mathbf{M_2} = \left\{\begin{matrix} \# & \# & \# \\ \# & S & \# \end{matrix}\right\}$;

$$\mathbf{C} = \left\{\begin{matrix} \# \\ \alpha \end{matrix} \middle| S \to \alpha \in P\right\}$$
$$\cup \left\{\begin{matrix} \# \\ S \end{matrix} \middle| S \to \alpha \in P\right\}$$
$$\cup \left\{\begin{matrix} \# \\ \gamma \end{matrix} \middle| \beta \to \gamma \in P, \beta \in N, \gamma \in (N \cup T)^*\right\}$$
$$\cup \left\{\begin{matrix} \# \\ \beta \end{matrix} \middle| \beta \to \gamma \in P, \beta \in N, \gamma \in (N \cup T)^*\right\};$$

$$\mathbf{R} = \left\{\begin{matrix} \# & c \\ \# & C_i \\ \# & f_i \end{matrix} \middle| C \to cC_if_i \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} \# & c \\ \# & C_i \end{matrix} \middle| C \to cC_i \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} \# & c \\ \# & \# \end{matrix} \middle| C \to c \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} c & d \\ C_i & D_j \\ f_i & f_j \end{matrix} \middle| C \to cC_if_i \in P_i, D \to dD_jf_j \in P_j, C \in N_i, D \in N_j\right\}$$
$$\cup \left\{\begin{matrix} c & d \\ C_i & D_j \end{matrix} \middle| C \to cC_i \in P_i, D \to dD_j \in Pj, C \in N_i, D \in N_j\right\}$$
$$\cup \left\{\begin{matrix} c & d \\ \# & \# \end{matrix} \middle| C \to c \in P_i, D \to d \in P_j\right\}$$
$$\cup \left\{\begin{matrix} c & \# \\ C_i & \# \\ f_i & \# \end{matrix} \middle| C \to cC_if_i \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} c & \# \\ C_i & \# \end{matrix} \middle| C \to cC_i \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} c & \# \\ \# & \# \end{matrix} \middle| C \to c \in P_i, C \in N_i\right\}$$
$$\cup \left\{\begin{matrix} c & d \\ C_i & D_j \end{matrix} \middle| Cf_i \to cC_i \in P_i, Df_j \to dD_j \in P_j\right\}$$
$$\cup \left\{\begin{matrix} \# & c \\ \# & C_i \end{matrix} \middle| Cf_i \to cC_i \in P_i\right\}$$
$$\cup \left\{\begin{matrix} c & \# \\ C_i & \# \end{matrix} \middle| Cf_i \to cC_i \in P_i\right\}$$
$$\cup \left\{\begin{matrix} \# & c \\ \# & \# \end{matrix} \middle| Cf_i \to c \in P_i\right\}$$

$$\cup\left\{\begin{matrix} c & d \\ \# & \# \end{matrix}\middle| Cf_i \to c \in P_i, Df_j \to d \in P_j\right\}$$
$$\cup\left\{\begin{matrix} c & \# \\ \# & \# \end{matrix}\middle| Cf_i \to c \in P_i\right\}$$
$$\cup\left\{\begin{matrix} \# & C \end{matrix}\middle| C \to cC_if_i \ or \ C \to cC_i \ or \ C \to c \in P_i\right\}$$
$$\cup\left\{\begin{matrix} C & D \end{matrix}\middle| C \to cC_if_i \ or \ C \to cC_i \ or \ C \to c \in P_i, D \to dD_jf_j \ or \ D \to dD_j \ or \ D \to d \in P_j\right\}$$
$$\cup\left\{\begin{matrix} C & \# \end{matrix}\middle| C \to cC_i \ or \ C \to c \in P_i\right\}$$
$$\cup\left\{\begin{matrix} C & D \\ f_i & f_j \end{matrix}\middle| Cf_i \to cC_i \ or \ Cf_i \to c \in P_i, Df_j \to dD_j \in P_j\right\}$$
$$\cup\left\{\begin{matrix} \# & C \\ \# & f_i \end{matrix}\middle| Cf_i \to cC_i \in P_i \ or \ Cf_i \to c \in P_i\right\}$$
$$\cup\left\{\begin{matrix} C & \# \\ f_i & \# \end{matrix}\middle| Cf_i \to cC_i \in P_i \ or \ Cf_i \to c \in P_i\right\}$$
$$\cup\left\{\begin{matrix} \# & \# \end{matrix}\right\}.$$

Corresponding to every production rule in P of the grammar G_1, we define insertion rules of $\mathbf{I_2}$ as follows:

(IR1) $$\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} \# & , & \# \\ \# & , & S \end{bmatrix} = \left\{\begin{matrix} \# \\ \alpha \end{matrix}\middle| S \to \alpha \in P\right\}, here\right\},\right.$$
$$\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} \# & , & \# \\ \# & , & \beta \end{bmatrix} = \left\{\begin{matrix} \# \\ \gamma \end{matrix}\middle| \beta \to \gamma \in P\right\}, here\right\},\right.$$
$$\left.\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{c}}\begin{bmatrix} \# & , & \# \\ A & , & \beta \end{bmatrix} = \left\{\begin{matrix} \# \\ \gamma \end{matrix}\middle| \beta \to \gamma \in P, A \in N \cup I\right\}, here\right\}\right.;$$

Corresponding to every production rule in P_i of the grammar $G_{2i} \in G_2$, we define insertion rules of $\mathbf{I_2}$ as follows:

(IR2) $$\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad \# \quad C] = \left\{\begin{matrix} \# & c \\ \# & C_i \\ \# & f_i \end{matrix}\middle| C \to cC_if_i \in P_i, C \in I, C_i \in N_i\right\},\right.\right.$$
$$\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad C \quad D] = \left\{\begin{matrix} c & d \\ C_i & D_j \\ f_i & g_j \end{matrix}\middle| C \to cC_if_i \in P_i, C \in I, C_i \in N_i, D \to dD_jf_j \in P_j, D \in I, D_j \in N_j\right\},$$
$$\left.\left.\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad C \quad \#] = \left\{\begin{matrix} c & \# \\ C_i & \# \\ f_i & \# \end{matrix}\middle| C \to cC_if_i \in P_i, C \in I, C_i \in N_i,\right\}\right\}, here\right\};$$

(IR3) $$\left\{\left\{\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad \# \quad C] = \left\{\begin{matrix} \# & c \\ \# & C_i \end{matrix}\middle| C \to cC_i \in P_i, C \in I, C_i \in N_i\right\},\right.\right.$$
$$\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad C \quad D] = \left\{\begin{matrix} c & d \\ C_i & D_j \end{matrix}\middle| C \to cC_i \in P_i, C \in I, C_i \in N_i, D \to dD_j \in P_j, D \in I, D_j \in N_j\right\},$$
$$\left.\left.\varphi^{\mathbf{i}}_{\mathbf{r}}[\# \quad \# \quad , \quad C \quad \#] = \left\{\begin{matrix} c & \# \\ C_i & \# \end{matrix}\middle| C \to cC_i \in P_i, C \in I, C_i \in N_i\right\}\right\}, here\right\};$$

(IR4) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad \# \quad C] = \left\{\begin{matrix} \# & c \\ \# & \# \end{matrix} \middle| Cf_i \to c \in P_i, C \in I, c \in T \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad C \quad D] = \left\{\begin{matrix} c & d \\ \# & \# \end{matrix} \middle| Cf_i \to c \in P_i, Dg_i \to d \in P_j, c, d \in T, C, D \in I \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad C \quad \#] = \left\{\begin{matrix} c & \# \\ \# & \# \end{matrix} \middle| Cf_i \to c \in P_i, C \in I, c \in T, \right\}\Bigg\}, here\Bigg\};$$

(IR5) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad \# \quad C] = \left\{\begin{matrix} \# & c \\ \# & \# \end{matrix} \middle| C \to c \in P_i, C \in I, c \in T \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad C \quad D] = \left\{\begin{matrix} c & d \\ \# & \# \end{matrix} \middle| C \to c \in P_i, c \in T, C \in I, D \to d \in P_j, d \in T, D \in I \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad \# \quad , \quad C \quad \#] = \left\{\begin{matrix} c & \# \\ \# & \# \end{matrix} \middle| C \to c \in P_i, C \in I, c \in T \right\}\Bigg\}, here\Bigg\};$$

(IR6) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}\begin{bmatrix} a & b & & C & D \\ c & d & , & f_i & f_j \end{bmatrix} = \left\{\begin{matrix} c & d \\ C_i & D_j \end{matrix} \middle| Cf_i \to cC_i \in P_i, a, b, c, d \in T, C \in I, C_i \in N_i, Df_j \to dD_j \in P_j, D \in I, D_j \in N_j \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}\begin{bmatrix} \# & a & & \# & C \\ \# & b, & & \# & f_i \end{bmatrix} = \left\{\begin{matrix} \# & c \\ \# & C_i \end{matrix} \middle| Cf_i \to cC_i \in P_i, a, b \in T, C \in I, C_i \in N_i \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}\begin{bmatrix} a & \# & & C & \# \\ b & \#, & & f_i & \# \end{bmatrix} = \left\{\begin{matrix} c & \# \\ C_i & \# \end{matrix} \middle| Cf_i \to cC_i \in P_i, a, b \in T, C \in I, C_i \in N_i \right\}\Bigg\}, here\Bigg\};$$

(IR7) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad a \quad , \quad \# \quad E] = \left\{\begin{matrix} \# & e \\ \# & \# \end{matrix} \middle| E \to e \in P_i, a, e \in T, E \in N_i \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[a \quad b \quad , \quad E \quad F] = \left\{\begin{matrix} e & f \\ \# & \# \end{matrix} \middle| E \to e \in P_i, a, e, f \in T, E \in N_i, F \in N_j \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[a \quad \# \quad , \quad E \quad \#] = \left\{\begin{matrix} e & \# \\ \# & \# \end{matrix} \middle| E \to e \in P_i, a, e \in T, E \in N_i \right\}\Bigg\} here\Bigg\};$$

(IR8) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad a \quad , \quad \# \quad E] = \left\{\begin{matrix} \# & e \\ \# & E_i \\ \# & f_i \end{matrix} \middle| E \to eE_if_i \in P_i, E, E_i \in N_i, e \in T \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[a \quad b \quad , \quad E \quad F] = \left\{\begin{matrix} e & f \\ E_i & F_j \\ f_i & f_j \end{matrix} \middle| E \to eE_if_i \in P_i, E, E_i \in N_i, e, f \in T, F \to fF_jf_j \in P_j, F, F_j \in N_j \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[a \quad \# \quad , \quad E \quad \#] = \left\{\begin{matrix} e & \# \\ E_i & \# \\ f_i & \# \end{matrix} \middle| E \to eE_if_i \in P_i, E, E_i \in N_i, e \in T \right\}\Bigg\}, here\Bigg\};$$

(IR9) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}[\# \quad a \quad , \quad \# \quad E] = \left\{\begin{matrix} \# & e \\ \# & E_i \end{matrix} \middle| E \to eE_i \in P_i, E, E_i \in N_i, e \in T \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}[a \quad b \quad , \quad E \quad F] = \left\{\begin{matrix} e & f \\ E_i & F_j \end{matrix} \middle| E \to eE_i \in P_i, E, E_i \in N_i, e, f \in T, F \to fF_j \in N_j, F, F_j \in N_j \right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}\left[a \quad \# \quad , \quad E \quad \#\right] = \left\{\begin{matrix} e & \# \\ E_i & \# \end{matrix} \middle| E \to eE_i \in P_i, E, E_i \in N_i, e \in T\right\}\Bigg\}, here\Bigg\}\Bigg\};$$

(IR10) $$\Bigg(\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}\left[\# \quad a \quad , \quad \# \quad E\right] = \left\{\begin{matrix} \# & e \\ \# & \# \end{matrix} \middle| E \to e \in P_i, a, e \in T, E \in N_i\right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}\left[a \quad b \quad , \quad E \quad F\right] = \left\{\begin{matrix} e & f \\ \# & \# \end{matrix} \middle| E \to e \in P_i, F \to f \in P_j, a, e, f \in T, E \in N_i, F \in N_j\right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{i}}\left[a \quad \# \quad , \quad E \quad \#\right] = \left\{\begin{matrix} e & \# \\ \# & \# \end{matrix} \middle| E \to e \in P_i, a, e \in T, E \in N_i\right\}\Bigg\}, here\Bigg\};$$

(IR11) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{i}}\left[a \quad b \quad , \quad \lambda \quad \lambda\right] = \left\{\lambda \quad \lambda \middle| a, b \in T\right\}\Bigg\}, out\Bigg\}.$$

Corresponding to every production rule in P of the grammar G_1, we define deletion rules of $\mathbf{D_2}$ as follows:

(DR1) $$\Bigg\{\Bigg\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} \# & & \# \\ \alpha & , & \# \end{bmatrix} = \left\{\begin{matrix} \# \\ s \end{matrix} \middle| s \to \alpha \in P\right\}, here\Bigg\},$$

$$\Bigg\{\Bigg\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} \# & & \# \\ \gamma & , & \# \end{bmatrix} = \left\{\begin{matrix} \# \\ \beta \end{matrix} \middle| \beta \to \gamma \in P\right\}, here\Bigg\},$$

$$\Bigg\{\Bigg\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} \# & & \# \\ \gamma & , & \# \end{bmatrix} = \left\{\begin{matrix} \# \\ B \end{matrix} \middle| \beta \to \gamma \in P, B \in N \cup I\right\}, here\Bigg\};$$

Corresponding to every production rule in P_i of the grammar $G_{2i} \in G_2$, we define deletion rules of $\mathbf{D_2}$ as follows:

(DR2) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & c & & & \\ \# & C_i & , & \lambda & \lambda \\ \# & f_i & & & \end{bmatrix} = \left\{\# \quad C \middle| C \to cC_if_i \in P_i, C_i \in N_i, C \in I\right\},$$

$$\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & d & & & \\ C_i & D_j & , & \lambda & \lambda \\ f_i & g_i & & & \end{bmatrix} = \left\{C \quad D \middle| C \to cC_if_i \in P_i, C_i \in N_i, C \in I, D \to dD_jf_j \in P_j, D_j \in N_j, D \in I\right\},$$

$$\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & \# & & & \\ C_i & \# & , & \lambda & \lambda \\ f_i & \# & & & \end{bmatrix} = \left\{C \quad \# \middle| C \to cC_if_i \in P_i, C_i \in N_i, C \in I\right\}\Bigg\}, here\Bigg\};$$

(DR3) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & c & & & \\ \# & C_i & , & \lambda & \lambda \end{bmatrix} = \left\{\# \quad C \middle| C \to cC_i \in P_i, C_i \in N_i, C \in I\right\},$$

$$\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & d & & & \\ C_i & D_j & , & \lambda & \lambda \end{bmatrix} = \left\{C \quad D \middle| C \to cC_i \in P_i, C_i \in N_i, C \in I, D \to dD_j \in P_j, D_j \in N_j, D \in I\right\},$$

$$\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & \# & & & \\ C_i & \# & , & \lambda & \lambda \end{bmatrix} = \left\{C \quad \# \middle| C \to cC_i \in P_i, C_i \in N_i, C \in I\right\}\Bigg\}, here\Bigg\};$$

(DR4) $$\Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & c & & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \left\{\# \quad C \middle| C \to c \in P_i, C \in I, c \in T\right\},$$

$$\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & d & & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \Bigg\{C \quad D \Bigg| C \to c \in P_i, c \in T, C \in I, D \to d \in P_j, d \in T,$$

$D \in I\Big\}, \Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & \# & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \Big\{C \quad \# \Big| C \to c \in P_i, c \in T, C \in I\Big\}\Big\}, here\Big\};$

(DR5) $\Big\{\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & c & & \\ \# & C_i & , & \lambda & \lambda \end{bmatrix} = \begin{Bmatrix} \# & C \\ \# & f_i \end{Bmatrix}\Big| Cf_i \to cC_i \in P_i, a, b \in T, C \in I, C_i \in N_i\Big\},$

$\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & d & & \\ C_i & D_j & , & \lambda & \lambda \end{bmatrix} = \begin{Bmatrix} C & D \\ f_i & f_j \end{Bmatrix}\Big| Cf_i \to cC_i \in P_i, C \in I, C_i \in N_i, Df_j \to dD_j \in P_j, D \in I, D_j \in N_j\Big\},$

$\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} c & \# & & \\ C_i & \# & , & \lambda & \lambda \end{bmatrix} = \begin{Bmatrix} C & \# \\ f_i & \# \end{Bmatrix}\Big| Cf_i \to cC_i \in P_i, c \in T, C \in I, C_i \in N_i\Big\}\Big\}, here\Big\};$

(DR6) $\Big\{\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & e & & \\ \# & E_i & , & \# & g_i \end{bmatrix} = \Big\{\# \quad E\Big| E \to eE_i \in P_i,\ E, E_i \in N_i, e \in T, g_i \in \mathcal{F}_i\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} e & f & & \\ E_i & F_j & , & g_i & g_j \end{bmatrix} = \Big\{E \quad F\Big| E \to eE_i \in P_i, E, E_i \in N_i, g_i \in \mathcal{F}_i, F \to fF_j \in P_j, F, F_j \in N_j, g_j \in \mathcal{F}_j, e, f \in T\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} e & \# & & \\ E_i & \# & , & g_i & \# \end{bmatrix} = \Big\{E \quad \#\Big| E \to eE_i \in P_i, E, E_i \in N_i, e \in T, g_i \in \mathcal{F}_i\Big\}\Big\}, here\Big\};$

(DR7) $\Big\{\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} \# & e & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \Big\{\# \quad E\Big| E \to e \in P_i, a, e \in T, E \in N_i\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} e & f & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \Big\{E \quad \#\Big| E \to e \in P_i, a, e, f \in T, E \in N_i, F \in N_j\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix} e & \# & & \\ \# & \# & , & \lambda & \lambda \end{bmatrix} = \Big\{E \quad \#\Big| E \to e \in P_i, a, e \in T, E \in N_i\Big\}\Big\}, here\Big\};$

(DR8) $\Big\{\Big\{\varphi_{\mathbf{r}}^{\mathbf{d}}[\# \quad a \quad , \quad \lambda \quad \lambda] = \Big\{\# \quad \#\Big| a \in T\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}[a \quad b \quad , \quad \lambda \quad \lambda] = \Big\{\# \quad \#\Big| a, b \in T\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}[a \quad \# \quad , \quad \lambda \quad \lambda] = \Big\{\# \quad \#\Big| a \in T\Big\}\Big\}, here\Big\};$

(DR9) $\Big\{\Big\{\varphi_{r}^{d}[\lambda \quad \lambda \quad , \quad \# \quad a] = \Big\{\# \quad \#\Big| a \in T\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}[\lambda \quad \lambda \quad , \quad a \quad b] = \Big\{\# \quad \#\Big| a, b \in T\Big\},$

$\varphi_{\mathbf{r}}^{\mathbf{d}}[\lambda \quad \lambda \quad , \quad a \quad \#] = \Big\{\# \quad \#\Big| a \in T\Big\}\Big\}, here\Big\};$

(DR10) $\Big\{\Big\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} \lambda & & a \\ \lambda & , & b \end{bmatrix} = \begin{Bmatrix} \# \\ \# \end{Bmatrix}\Big| a, b \in T\Big\}\Big\}, here\Big\},$

$\Big\{\Big\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} a & & \lambda \\ b & , & \lambda \end{bmatrix} = \begin{Bmatrix} \# \\ \# \end{Bmatrix}\Big| a, b \in T\Big\}\Big\}, here\Big\},$

$\Big\{\Big\{\varphi_{\mathbf{c}}^{\mathbf{d}}\begin{bmatrix} a & & \lambda \\ b & , & \lambda \end{bmatrix} = \begin{Bmatrix} \# \\ \# \end{Bmatrix}\Big| a, b \in T\Big\}\Big\}, here\Big\};$

$$\text{(DR11)}\quad \Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}\# & c & & & \\ \# & C_i & , & \# & g_i\end{bmatrix} = \left\{\begin{matrix}\# & C\\ \# & f_i\end{matrix}\,\middle|\, Cf_i \to cC_i \in P_i, g_i \in \mathcal{F}_i, C \in I, C_i \in N_i\right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}c & d & & & \\ C_i & D_j & , & g_i & g_j\end{bmatrix} = \Bigg\{\begin{matrix}C & D\\ f_i & f_j\end{matrix}\,\Bigg|\, Cf_i \to cC_i \in P_i, g_i \in \mathcal{F}_i, C \in I, C_i \in N_i, Df_j \to$$

$$dD_j \in P_j, g_j \in \mathcal{F}_j, D_j \in N_j, D \in I\Bigg\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}c & \# & & & \\ C_i & \# & , & g_i & \#\end{bmatrix} = \left\{\begin{matrix}C & \#\\ f_i & \#\end{matrix}\,\middle|\, Cf_i \to cC_i \in P_i, g_i \in \mathcal{F}_i, C \in I, C_i \in N_i\right\}\Bigg\}, here\Bigg\};$$

$$\text{(DR12)}\quad \Bigg\{\Bigg\{\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}\# & e & & & \\ \# & E_i & & \# & g_i\\ \# & f_i & , & & \end{bmatrix} = \left\{\begin{matrix}\# & E\end{matrix}\,\middle|\, E \to eE_if_i \in P_i, E, E_i \in N_i, g_i \in \mathcal{F}_i, e \in T\right\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}e & f & & & \\ E_i & F_j & & g_i & g_j\\ f_i & f_j & , & & \end{bmatrix} = \Bigg\{E \quad F\,\Bigg|\, E \to eE_if_i \in P_i, E, E_i \in N_i, e, f \in T, g_i \in \mathcal{F}_i, F \to$$

$$fF_jf_j \in P_j, F, F_j \in N_j, g_j \in \mathcal{F}_j\Bigg\},$$

$$\varphi_{\mathbf{r}}^{\mathbf{d}}\begin{bmatrix}e & \# & & & \\ E_i & \# & & g_i & \#\\ f_i & \# & , & & \end{bmatrix} = \Bigg\{E \quad \#\,\Bigg|\, E \to eE_if_i \in P_i, E, E_i \in N_i, e \in T, g_i \in$$

$$\mathcal{F}_i\Bigg\}\Bigg\}, here\Bigg\}.$$

In membrane 2, there are eleven sets of column and row insertion rules of $\mathbf{I_2}$ grouped according to the rules of G_1 and G_2 respectively of the grammar G.

For every rule $S \to \alpha \in P$ in G_1, column insertion rule(s) to insert the context $\substack{\#\\ \alpha}$ between $\substack{\#\\ \#}$ and $\substack{\#\\ S}$ are defined.

For every rule of the form $\beta \to \gamma \in P$ in G_1 the corresponding column insertion rules are defined to insert the context $\substack{\#\\ \gamma}$ between $\substack{\#\\ \#}$ and $\substack{\#\\ \beta}$ (or) $\substack{\#\\ A}$ and $\substack{\#\\ \beta}$.

In membrane 2, there are twelve sets of column and row deletion rules defined in $\mathbf{D_2}$ based on the rules of G_1 and G_2 respectively of the grammar G.

For every rule $S \to \alpha \in P$ in G_1 deletion rule(s) to delete the contexts $\substack{\#\\ S}$ between $\substack{\#\\ \alpha}$ and $\substack{\#\\ \#}$ are defined.

For every rule of the form $\beta \to \gamma \in P$ in G_1 the corresponding column deletion rules are defined to delete the context $\substack{\#\\ \beta}$ between $\substack{\#\\ \alpha}$ and $\substack{\#\\ \#}$ (or) $\substack{\#\\ \gamma}$ and $\substack{\#\\ \#}$.

Using these column insertion and column deletion rules of the P-system the same horizontal derivation of the horizontal grammar G_1 of G can be achieved. It is to be noted that the rules for the horizontal growth in I_2 and D_2 of the P-system are defined in such a manner that the column insertion rules and column deletion rules are applied alternatively.

To simulate the vertical derivation of the grammar G, row insertion and row deletion rules are defined based on the rules of the vertical grammars $G_{2i}(1 \leq i \leq k)$ of G_2 in G.

For the rules of the form $C \to cC_if_i$, $C \to cC_i$, $C \to c$, $Cf_i \to cC_i$ in P_i of G_{2i} the corresponding row insertion and row deletion rules are defined in $\mathbf{I_2}$ and $\mathbf{D_2}$ respectively to replicate the vertical derivation of the vertical grammars G_{2i}.

It should be noted that we represent, for example $\frac{\#}{a}, \frac{\#}{b}, \frac{\#}{c}, \frac{\#}{d}$ as $\frac{\#}{\alpha}$ where $\alpha \in \{a, b, c, d\}$.

It is again to be noted that the rules for vertical growth in $\mathbf{I_2}$ and $\mathbf{D_2}$ of the P-system are defined in such a manner that the row insertion and row deletion rules are applied alternatively. A row insertion rule in $\mathbf{I_2}$ is defined to send the generated picture to the skin membrane which is the output membrane.

The working of the P-system in membrane 2 is as follows: The axiom set consists of the array $\begin{matrix}\# & \# & \#\\ \# & S & \#\end{matrix}$ based on the starting symbol S of any $(CF:RIR)SMG$ G. We consider the rules in $\mathbf{I_2}$ and $\mathbf{D_2}$ to perform the parallel contextual column insertion and column deletion operations and these operations are performed alternatively to simulate the generation of horizontal strings of intermediates based on G_1 of G. Now we consider the rules in $\mathbf{I_2}$ and $\mathbf{D_2}$ to perform the parallel contextual row insertion operation and row deletion operations. Parallel contextual row insertion and deletion operations are performed alternatively to simulate the vertical generation of the picture based on G_{2i}. Then using the parallel contextual column deletion rules in $\mathbf{D_2}$, the #'s along the borders of the columns are deleted and finally using the parallel contextual row deletion rules in $\mathbf{D_2}$ the #'s along the borders of the rows are deleted. The resulting arrays belong to $L(G)$. Finally, using the rule $\left(\left\{\varphi_r^i\left[a \quad b \quad , \quad \lambda \quad \lambda\right] = \left\{\lambda \quad \lambda \,\middle|\, a, b \in T\right\}\right\}, out\right) \in \mathbf{I_2}$, the resultant arrays are sent to membrane 1, which is the output membrane.

We now prove that the inclusion is proper. James et al [15] have proved that the context-sensitive matrix language

$$L = \left\{ \begin{pmatrix} (\bullet)_n \\ (X)_n \\ (\bullet)_n \end{pmatrix}^n \begin{pmatrix} (X)_n \\ (X)_n \\ (\bullet)_n \end{pmatrix}^n \begin{pmatrix} (\bullet)_n \\ (\bullet)_n \\ (\bullet)_n \end{pmatrix}^n \Big/ n \geq 1 \right\}$$

can be generated by a $PCAIDPS_2$. It is clear that L cannot be generated by any $(CF : RIR)SMG$ as the horiztontal production rules of $(CF : RIR)SMG$ are at the best context-free. Hence $\mathfrak{L}(CF : RIR)$ is properly included in the family $\mathfrak{L}(PCAIDPS_2)$. □

The verification of the proof of the **Theorem 3.4** using **Example 3.3** is given in the appendix A.

4. Conclusion

We have the following obsertions:

(i) The arrays generated by PCAIDPS are collected in the membrane 1 considered as the output membrane. Since the membrane 1 serves the purpose of collection of the arrays, the PCAIDPS in **Theorem 3.4** can be considered as a P system with one membrane if the arrays generated by the system is the same as arrays sent out of the membrane system.

(ii) In one of the pictures generated by $PCAIDPS_2$ in **Example 2.4**, if we replace the terminal X by the primitive $\sqsubset\!\sqsupset$ and all $\bullet$ by blank space, we get the following image

$$\begin{array}{cccc} \sqsubset\!\sqsupset & & & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & & & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & \sqsubset\!\sqsupset & \sqsubset\!\sqsupset & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & & & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & & & \sqsubset\!\sqsupset \end{array}$$

and if we replace X's by the primitive of the form $\sqsubset\!\sqsupset$, we get the following image

$$\begin{array}{cccc} \sqsubset\!\sqsupset & \bullet & \bullet & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & \bullet & \bullet & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & \sqsubset\!\sqsupset & \sqsubset\!\sqsupset & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & \bullet & \bullet & \sqsubset\!\sqsupset \\ \sqsubset\!\sqsupset & \bullet & \bullet & \sqsubset\!\sqsupset \end{array} .$$

(iii) PCAIDPS has greater generative capacity to obtain some known families of two dimensional array languages. So far, we could show that $\mathfrak{L}(PCAIDPS_2)$ includes REC, $\mathfrak{L}(CSMG)$ and $\mathfrak{L}(CF : RIR)$. We note that these three classes are incomparable but not disjoint. It is worth investigating to form a hierarchy among the families of two dimensional array languages. This paper strives in this direction.

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Appendix A. Verification of the proof of the Theorem 3.4 using Example 3.3

We have listed only the insertion and deletion rules of the P systems $\prod$ corresponding to the production rules of the $(CF : RIR)SMG$ of *Example* 3.3 as per the proof of *Theorem* 3.4.

It should be noted that the rules of the P system $\prod$ are listed and grouped in order to simulate the behaviour of the successful derivations of the $(CF : RIR)SMG$ G of *Example* 3.3 with the help of the axiom array $\begin{matrix} \# & \# & \# \\ \# & S & \# \end{matrix}$ in M_2 of $\prod$, omitting the useless rules.

Consider $G_1 = \{S \to S_1A, A \to S_2A, A \to S_2S_1\}$.

For $S \to S_1A$, the corresponding column insertion rule of the P system is

$$\text{(IR1)} \quad \left\{ \varphi_c^i \begin{bmatrix} \# & & \# \\ \# & , & S \end{bmatrix} = \begin{matrix} \# & \# \\ S_1 & A \end{matrix}, \ here \right\}.$$

For $A \to S_2A$, we have

$$\text{(IR2)} \quad \left\{ \left\{ \varphi_c^i \begin{bmatrix} \# & & \# \\ S_1 & , & A \end{bmatrix} = \begin{matrix} \# & \# \\ S_2 & A \end{matrix}, \ \varphi_c^i \begin{bmatrix} \# & & \# \\ S_2 & , & A \end{bmatrix} = \begin{matrix} \# & \# \\ S_2 & A \end{matrix} \right\}, \ here \right\}.$$

For $A \to S_2S_1$, we have

$$\text{(IR3)} \quad \left\{ \left\{ \varphi_c^i \begin{bmatrix} \# & & \# \\ S_1 & , & A \end{bmatrix} = \begin{matrix} \# & \# \\ S_2 & S_1 \end{matrix}, \ \varphi_c^i \begin{bmatrix} \# & & \# \\ S_2 & , & A \end{bmatrix} = \begin{matrix} \# & \# \\ S_2 & S_1 \end{matrix} \right\}, \ here \right\},$$

Now Consider G_2.

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_1f_2 \in P_{22}$, we have

$$\text{(IR4)} \quad \left\{ \left\{ \varphi_r^i \left[\# \ \ \# \ \ , \ \ \# \ \ S_1 \right] = \begin{matrix} \# & x \\ \# & A_1 \\ \# & g_2 \end{matrix}, \ \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_1 \ \ S_2 \right] = \begin{matrix} x & \bullet \\ A_1 & B_1 \\ g_2 & f_2 \end{matrix}, \right.\right.$$

$$\varphi_r^i \left[\# \ \ \# \ \ , \ \ S_2 \ \ S_2 \right] = \begin{matrix} \bullet & \bullet \\ B_1 & B_1 \\ f_2 & f_2 \end{matrix}, \ \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_2 \ \ S_1 \right] = \begin{matrix} \bullet & x \\ B_1 & A_1 \\ f_2 & g_2 \end{matrix},$$

$$\left.\left. \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_1 \ \ \# \right] = \begin{matrix} x & \# \\ A_1 & \# \\ g_2 & \# \end{matrix} \right\}, \ here \right\}.$$

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_2f_2 \in P_{22}$, we have

$$\text{(IR5)} \quad \left\{ \left\{ \varphi_r^i \left[\# \ \ \# \ \ , \ \ \# \ \ S_1 \right] = \begin{matrix} \# & x \\ \# & A_1 \\ \# & g_2 \end{matrix}, \ \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_1 \ \ S_2 \right] = \begin{matrix} x & \bullet \\ A_1 & B_2 \\ g_2 & f_2 \end{matrix}, \right.\right.$$

$$\varphi_r^i \left[\# \ \ \# \ \ , \ \ S_2 \ \ S_2 \right] = \begin{matrix} \bullet & \bullet \\ B_2 & B_2 \\ f_2 & f_2 \end{matrix}, \ \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_2 \ \ S_1 \right] = \begin{matrix} \bullet & x \\ B_2 & A_1 \\ f_2 & g_2 \end{matrix},$$

$$\left.\left. \varphi_r^i \left[\# \ \ \# \ \ , \ \ S_1 \ \ \# \right] = \begin{matrix} x & \# \\ A_1 & \# \\ g_2 & \# \end{matrix} \right\}, \ here \right\}.$$

For $A_1 \to xA_1g_1 \in P_{21}$ and $B_1 \to \bullet B_1f_1 \in P_{22}$, we have

$$\text{(IR6)} \quad \left\{ \left\{ \varphi_r^i \left[\# \ \ x \ \ , \ \ \# \ \ A_1 \right] = \begin{matrix} \# & x \\ \# & A_1 \\ \# & g_1 \end{matrix}, \ \varphi_r^i \left[x \ \ \bullet \ \ , \ \ A_1 \ \ B_1 \right] = \begin{matrix} x & \bullet \\ A_1 & B_1 \\ g_1 & f_1 \end{matrix}, \right.\right.$$

$$\varphi_r^i\left[\bullet \quad \bullet \quad , \quad B_1 \quad B_1\right] = \begin{matrix} \bullet & \bullet \\ B_1 & B_1 \\ f_1 & f_1 \end{matrix}, \quad \varphi_r^i\left[\bullet \quad x \quad , \quad B_1 \quad A_1\right] = \begin{matrix} \bullet & x \\ B_1 & A_1 \\ f_1 & g_1 \end{matrix},$$

$$\varphi_r^i\left[x \quad \# \quad , \quad A_1 \quad \#\right] = \begin{matrix} x & \# \\ A_1 & \# \\ g_1 & \# \end{matrix}\Bigg\}, \ here\Bigg\}.$$

For $A_1 \to xA_1g_1 \in P_{21}$ and $B_1 \to \bullet B_2 f_1 \in P_{22}$, we have

$$\text{(IR7)} \quad \Bigg\{\Bigg\{\varphi_r^i\left[\# \quad x \quad , \quad \# \quad A_1\right] = \begin{matrix} \# & x \\ \# & A_1 \\ \# & g_1 \end{matrix}, \quad \varphi_r^i\left[x \quad \bullet \quad , \quad A_1 \quad B_1\right] = \begin{matrix} x & \bullet \\ A_1 & B_2 \\ g_1 & f_1 \end{matrix},$$

$$\varphi_r^i\left[\bullet \quad \bullet \quad , \quad B_1 \quad B_1\right] = \begin{matrix} \bullet & \bullet \\ B_2 & B_2 \\ f_1 & f_1 \end{matrix}, \quad \varphi_r^i\left[\bullet \quad x \quad , \quad B_1 \quad A_1\right] = \begin{matrix} \bullet & x \\ B_2 & A_1 \\ f_1 & g_1 \end{matrix},$$

$$\varphi_r^i\left[x \quad \# \quad , \quad A_1 \quad \#\right] = \begin{matrix} x & \# \\ A_1 & \# \\ g_1 & \# \end{matrix}\Bigg\}, \ here\Bigg\}.$$

For $A_1 \to A_2g_1 \in P_{21}$ and $B_2 \to B_3f_1 \in P_{22}$, we have

$$\text{(IR8)} \quad \Bigg\{\Bigg\{\varphi_r^i\left[\# \quad x \quad , \quad \# \quad A_1\right] = \begin{matrix} \# & A_2 \\ \# & g_1 \end{matrix}, \quad \varphi_r^i\left[x \quad \bullet \quad , \quad A_1 \quad B_2\right] = \begin{matrix} A_2 & B_3 \\ g_1 & f_1 \end{matrix},$$

$$\varphi_r^i\left[\bullet \quad \bullet \quad , \quad B_2 \quad B_2\right] = \begin{matrix} B_3 & B_3 \\ f_1 & f_1 \end{matrix}, \quad \varphi_r^i\left[\bullet \quad x \quad , \quad B_2 \quad A_1\right] = \begin{matrix} B_3 & A_2 \\ f_1 & g_2 \end{matrix},$$

$$\varphi_r^i\left[x \quad \# \quad , \quad A_1 \quad \#\right] = \begin{matrix} A_2 & \# \\ g_1 & \# \end{matrix}\Bigg\}, \ here\Bigg\}.$$

For $A_2g_1 \to xA_3 \in P_{21}$ and $B_3f_1 \to xB_4 \in P_{22}$, we have

$$\text{(IR9)} \quad \Bigg\{\Bigg\{\varphi_r^i\begin{bmatrix} \# & x & , & \# & A_2 \\ & & & \# & g_1 \end{bmatrix} = \begin{matrix} \# & x \\ \# & A_3 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} x & \bullet & , & A_2 & B_3 \\ & & & g_1 & f_1 \end{bmatrix} = \begin{matrix} x & x \\ A_3 & B_4 \end{matrix},$$

$$\varphi_r^i\begin{bmatrix} \bullet & \bullet & , & B_3 & B_3 \\ & & & f_1 & f_1 \end{bmatrix} = \begin{matrix} x & x \\ B_4 & B_4 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} \bullet & x & , & B_3 & A_2 \\ & & & f_1 & g_1 \end{bmatrix} = \begin{matrix} x & x \\ B_4 & A_3 \end{matrix},$$

$$\varphi_r^i\begin{bmatrix} x & \# & , & A_2 & \# \\ & & & g_1 & \# \end{bmatrix} = \begin{matrix} x & \# \\ A_3 & \# \end{matrix}\Bigg\}, \ here\Bigg\}.$$

For $A_3g_1 \to xA_3 \in P_{21}$ and $B_4f_1 \to \bullet\ B_4 \in P_{22}$, we have

$$\text{(IR10)} \quad \Bigg\{\Bigg\{\varphi_r^i\begin{bmatrix} \# & x & , & \# & A_3 \\ & & & \# & g_1 \end{bmatrix} = \begin{matrix} \# & x \\ \# & A_3 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} x & x & , & A_3 & B_4 \\ & & & g_1 & f_1 \end{bmatrix} = \begin{matrix} x & \bullet \\ A_3 & B_4 \end{matrix},$$

$$\varphi_r^i\begin{bmatrix} x & x & , & B_4 & B_4 \\ & & & f_1 & f_1 \end{bmatrix} = \begin{matrix} \bullet & \bullet \\ B_4 & B_4 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} x & x & , & B_4 & A_3 \\ & & & f_1 & g_1 \end{bmatrix} = \begin{matrix} \bullet & x \\ B_4 & A_3 \end{matrix},$$

$$\varphi_r^i\begin{bmatrix} x & \bullet & , & A_3 & B_4 \\ & & & g_1 & f_1 \end{bmatrix} = \begin{matrix} x & \bullet \\ A_3 & B_4 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} \bullet & \bullet & , & B_4 & B_4 \\ & & & f_1 & f_1 \end{bmatrix} = \begin{matrix} \bullet & \bullet \\ B_4 & B_4 \end{matrix},$$

$$\varphi_r^i\begin{bmatrix} \bullet & x & , & B_4 & A_3 \\ & & & f_1 & g_1 \end{bmatrix} = \begin{matrix} \bullet & x \\ B_4 & A_3 \end{matrix}, \quad \varphi_r^i\begin{bmatrix} x & \# & , & A_3 & \# \\ & & & g_1 & \# \end{bmatrix} = \begin{matrix} x & \# \\ A_3 & \# \end{matrix}\Bigg\}, \ here\Bigg\}.$$

For $A_3g_2 \to x \in P_{21}$ and $B_4f_2 \to \bullet \in P_{22}$, we have

$$\text{(IR11)} \quad \Bigg\{\Bigg\{\varphi_r^i\begin{bmatrix} \# & x & , & \# & A_3 \\ & & & \# & g_2 \end{bmatrix} = \begin{matrix} \# & x \\ \# & \# \end{matrix}, \quad \varphi_r^i\begin{bmatrix} x & \bullet & , & A_3 & B_4 \\ & & & g_2 & f_2 \end{bmatrix} = \begin{matrix} x & \bullet \\ \# & \# \end{matrix},$$

$$\varphi_r^i \begin{bmatrix} \bullet & \bullet & , & B_4 & B_4 \\ & & & f_2 & f_2 \end{bmatrix} = \begin{matrix} \bullet & \bullet \\ \# & \# \end{matrix}, \ \varphi_r^i \begin{bmatrix} \bullet & x & , & B_4 & A_3 \\ & & & f_2 & g_2 \end{bmatrix} = \begin{matrix} \bullet & x \\ \# & \# \end{matrix},$$

$$\left.\left.\varphi_r^i \begin{bmatrix} x & \# & , & A_3 & \# \\ & & & g_2 & \# \end{bmatrix} = \begin{matrix} x & \# \\ \# & \# \end{matrix} \right\}, \ here \right\}.$$

Consider again G_1.

For $S \to S_1 A$, the corresponding column deletion rule of the P system is

(DR1) $$\left\{ \varphi_c^d \begin{bmatrix} \# & \# & & \# \\ S_1 & A & , & \# \end{bmatrix} = \begin{matrix} \# \\ S \end{matrix}, \ here \right\}.$$

For $A \to S_2 A$, we have

(DR2) $$\left\{ \left\{ \varphi_c^d \begin{bmatrix} \# & \# & & \# \\ S_2 & A & , & \# \end{bmatrix} = \begin{matrix} \# \\ A \end{matrix} \right\}, here \right\}.$$

For $A \to S_2 S_1$, we have

(DR3) $$\left\{ \left\{ \varphi_c^d \begin{bmatrix} \# & \# & & \# \\ S_2 & S_1 & , & \# \end{bmatrix} = \begin{matrix} \# \\ A \end{matrix} \right\}, \ here \right\}.$$

Consider again G_2.

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_1 f_2$ we have,

(DR4) $$\left\{ \left\{ \phi_r^d \begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \lambda & \lambda \\ \# & g_2 & & & \end{bmatrix} = \# \quad S_1, \ \phi_r^d \begin{bmatrix} x & \bullet & & & \\ A_1 & B_1 & , & \lambda & \lambda \\ g_2 & f_2 & & & \end{bmatrix} = S_1 \quad S_2, \right.\right.$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet & & & \\ B_1 & B_1 & , & \lambda & \lambda \\ f_2 & f_2 & & & \end{bmatrix} = S_2 \quad S_2, \ \phi_r^d \begin{bmatrix} \bullet & x & & & \\ B_1 & A_1 & , & \lambda & \lambda \\ f_2 & g_2 & & & \end{bmatrix} = S_2 \quad S_1,$$

$$\left.\left.\phi_r^d \begin{bmatrix} x & \# & & & \\ A_1 & \# & , & \lambda & \lambda \\ g_2 & \# & & & \end{bmatrix} = S_2 \quad \# \right\}, here \right\}.$$

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_2 f_2$ we have,

(DR5) $$\left\{ \left\{ \phi_r^d \begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \lambda & \lambda \\ \# & g_2 & & & \end{bmatrix} = \# \quad S_1, \ \phi_r^d \begin{bmatrix} x & \bullet & & & \\ A_1 & B_2 & , & \lambda & \lambda \\ g_2 & f_2 & & & \end{bmatrix} = S_1 \quad S_2, \right.\right.$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet & & & \\ B_2 & B_2 & , & \lambda & \lambda \\ f_2 & f_2 & & & \end{bmatrix} = S_2 \quad S_2, \ \phi_r^d \begin{bmatrix} \bullet & x & & & \\ B_2 & A_1 & , & \lambda & \lambda \\ f_2 & g_2 & & & \end{bmatrix} = S_2 \quad S_1,$$

$$\left.\left.\phi_r^d \begin{bmatrix} x & \# & & & \\ A_1 & \# & , & \lambda & \lambda \\ g_2 & \# & & & \end{bmatrix} = S_2 \quad \# \right\}, here \right\}.$$

For $A_1 \to xA_1g_1 \in P_{21}$ and $B_1 \to \bullet B_1 f_1$ we have,

(DR6) $$\left\{ \left\{ \phi_r^d \begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \# & g_2 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1, \ \phi_r^d \begin{bmatrix} x & \bullet & & & \\ A_1 & B_1 & , & g_2 & f_2 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_1, \right.\right.$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet & & & \\ B_1 & B_1 & , & f_2 & f_2 \\ f_1 & f_1 & & & \end{bmatrix} = B_1 \quad B_1, \ \phi_r^d \begin{bmatrix} \bullet & x & & & \\ B_1 & A_1 & , & f_2 & g_2 \\ f_1 & g_1 & & & \end{bmatrix} = B_1 \quad A_1,$$

$$\phi_r^d \begin{bmatrix} x & \# \\ A_1 & \# \\ g_1 & \# \end{bmatrix}, g_2 \quad \# \Bigg] = A_1 \quad \# \phi_r^d \begin{bmatrix} \# & x \\ \# & A_1 \\ \# & g_1 \end{bmatrix}, \# \quad g_1 \Bigg] = \# \quad A_1 ,$$

$$\phi_r^d \begin{bmatrix} x & \bullet \\ A_1 & B_1 \\ g_1 & f_1 \end{bmatrix}, g_1 \quad f_1 \Bigg] = A_1 \quad B_1 , \phi_r^d \begin{bmatrix} \bullet & \bullet \\ B_1 & B_1 \\ f_1 & f_1 \end{bmatrix}, f_1 \quad f_1 \Bigg] = B_1 \quad B_1 ,$$

$$\phi_r^d \begin{bmatrix} \bullet & x \\ B_1 & A_1 \\ f_1 & g_1 \end{bmatrix}, f_1 \quad g_1 \Bigg] = B_1 \quad A_1 , \phi_r^d \begin{bmatrix} x & \# \\ A_1 & \# \\ g_1 & \# \end{bmatrix}, g_1 \quad \# \Bigg] =$$

$$A_1 \quad \# \Bigg\}, here \Bigg\}.$$

For $A_1 \to xA_1g_1 \in P_{21}$ and $B_1 \to \bullet B_2 f_1$ we have

(DR7) $$\Bigg\{\Bigg\{ \phi_r^d \begin{bmatrix} \# & x \\ \# & A_1 \\ \# & g_1 \end{bmatrix}, \# \quad g_2 \Bigg] = \# \quad A_1 , \phi_r^d \begin{bmatrix} x & \bullet \\ A_1 & B_2 \\ g_1 & f_1 \end{bmatrix}, g_2 \quad f_2 \Bigg] = A_1 \quad B_1 ,$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet \\ B_2 & B_2 \\ f_1 & f_1 \end{bmatrix}, f_2 \quad f_2 \Bigg] = B_1 \quad B_1 , \phi_r^d \begin{bmatrix} \bullet & x \\ B_2 & A_1 \\ f_1 & g_1 \end{bmatrix}, f_2 \quad g_2 \Bigg] = B_1 \quad A_1 ,$$

$$\phi_r^d \begin{bmatrix} x & \# \\ A_1 & \# \\ g_1 & \# \end{bmatrix}, g_2 \quad \# \Bigg] = A_1 \quad \# \Bigg\}, here \Bigg\}.$$

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_1 f_2$ we will have

(DR8) $$\Bigg\{\Bigg\{ \phi_r^d \begin{bmatrix} \# & x \\ \# & A_1 \\ \# & g_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = \# \quad S_1 , \phi_r^d \begin{bmatrix} x & \bullet \\ A_1 & B_1 \\ g_2 & f_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_1 \quad S_2 ,$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet \\ B_1 & B_1 \\ f_2 & f_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_2 \quad S_2 , \phi_r^d \begin{bmatrix} \bullet & x \\ B_1 & A_1 \\ f_2 & g_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_2 \quad S_1 ,$$

$$\phi_r^d \begin{bmatrix} x & \# \\ A_1 & \# \\ g_2 & \# \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_1 \quad \# \Bigg\}, here \Bigg\}.$$

For $S_1 \to xA_1g_2 \in P_{21}$ and $S_2 \to \bullet B_2 f_2$ we will have

(DR9) $$\Bigg\{\Bigg\{ \phi_r^d \begin{bmatrix} \# & x \\ \# & A_1 \\ \# & g_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = \# \quad S_1 , \phi_r^d \begin{bmatrix} x & \bullet \\ A_1 & B_2 \\ g_2 & f_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_1 \quad S_2 ,$$

$$\phi_r^d \begin{bmatrix} \bullet & \bullet \\ B_2 & B_2 \\ f_2 & f_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_2 \quad S_2 , \phi_r^d \begin{bmatrix} \bullet & x \\ B_2 & A_1 \\ f_2 & g_2 \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_2 \quad S_1 ,$$

$$\phi_r^d \begin{bmatrix} x & \# \\ A_1 & \# \\ g_2 & \# \end{bmatrix}, \lambda \quad \lambda \Bigg] = S_1 \quad \# \Bigg\}, here \Bigg\}.$$

For $A_1 \to xA_1g_1$ and $B_1 \to \bullet B_1 f_1$ we will have

$$\text{(DR10)}\quad \Bigg(\Bigg\{\phi_r^d\begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \# & g_2 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,\ \phi_r^d\begin{bmatrix} x & \bullet & & & \\ A_1 & B_1 & , & g_2 & f_2 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_1\ ,$$

$$\phi_r^d\begin{bmatrix} \bullet & \bullet & & & \\ B_1 & B_1 & , & f_2 & f_2 \\ f_1 & f_1 & & & \end{bmatrix} = B_1 \quad B_1\ ,\ \phi_r^d\begin{bmatrix} \bullet & x & & & \\ B_1 & A_1 & , & f_2 & g_2 \\ f_1 & g_1 & & & \end{bmatrix} = B_1 \quad A_1\ ,$$

$$\phi_r^d\begin{bmatrix} x & \# & & & \\ A_1 & \# & , & g_2 & \# \\ g_1 & \# & & & \end{bmatrix} = A_1 \quad \#\ ,\ \phi_r^d\begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \# & g_1 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,$$

$$\phi_r^d\begin{bmatrix} x & \bullet & & & \\ A_1 & B_1 & , & g_1 & f_1 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_1\ ,\ \phi_r^d\begin{bmatrix} \bullet & \bullet & & & \\ B_1 & B_1 & , & f_1 & f_1 \\ f_1 & f_1 & & & \end{bmatrix} = B_1 \quad B_1\ ,$$

$$\phi_r^d\begin{bmatrix} \bullet & x & & & \\ B_1 & A_1 & , & f_1 & g_1 \\ f_1 & g_1 & & & \end{bmatrix} = B_1 \quad A_1\ ,\ \phi_r^d\begin{bmatrix} x & \# & & & \\ A_1 & \# & , & g_1 & \# \\ g_1 & \# & & & \end{bmatrix} =$$

$$A_1 \quad \#\Bigg\}, here\Bigg\}.$$

For $A_1 \to xA_1g_1 \in P_{21}$ and $B_1 \to \bullet B_2f_1 \in P_{22}$ we have

$$\text{(DR11)}\quad \Bigg\{\Bigg\{\phi_r^d\begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \# & g_2 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,\ \phi_r^d\begin{bmatrix} x & \bullet & & & \\ A_1 & B_2 & , & g_2 & f_2 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_1\ ,$$

$$\phi_r^d\begin{bmatrix} \bullet & \bullet & & & \\ B_2 & B_2 & , & f_2 & f_2 \\ f_1 & f_1 & & & \end{bmatrix} = B_1 \quad B_1\ ,\ \phi_r^d\begin{bmatrix} \bullet & x & & & \\ B_2 & A_1 & , & f_2 & g_2 \\ f_1 & g_1 & & & \end{bmatrix} = B_1 \quad A_1\ ,$$

$$\phi_r^d\begin{bmatrix} x & \# & & & \\ A_1 & \# & , & g_2 & \# \\ g_1 & \# & & & \end{bmatrix} = A_1 \quad \#\ ,\ \phi_r^d\begin{bmatrix} \# & x & & & \\ \# & A_1 & , & \# & g_1 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,$$

$$\phi_r^d\begin{bmatrix} x & \bullet & & & \\ A_1 & B_2 & , & g_1 & f_1 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_1\ ,\ \phi_r^d\begin{bmatrix} \bullet & \bullet & & & \\ B_2 & B_2 & , & f_1 & f_1 \\ f_1 & f_1 & & & \end{bmatrix} = B_1 \quad B_1\ ,$$

$$\phi_r^d\begin{bmatrix} \bullet & x & & & \\ B_2 & A_1 & , & f_1 & g_1 \\ f_1 & g_1 & & & \end{bmatrix} = B_1 \quad A_1\ ,\ \phi_r^d\begin{bmatrix} x & \# & & & \\ A_1 & \# & , & g_1 & \# \\ g_1 & \# & & & \end{bmatrix} =$$

$$A_1 \quad \#\Bigg\}, here\Bigg\}.$$

For $A_1 \to A_2g_1 \in P_{21}$ and $B_2 \to B_1f_1 \in P_{22}$ we have

$$\text{(DR12)}\quad \Bigg\{\Bigg\{\phi_r^d\begin{bmatrix} \# & A_2 & , & \# & g_1 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,\ \phi_r^d\begin{bmatrix} A_2 & B_3 & , & g_1 & f_1 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_2\ ,$$

$$\phi_r^d\begin{bmatrix} B_3 & B_3 & , & f_1 & f_1 \\ f_1 & f_1 & & & \end{bmatrix} = B_2 \quad B_2\ ,\ \phi_r^d\begin{bmatrix} A_2 & \# & , & g_1 & \# \\ g_1 & \# & & & \end{bmatrix} = A_1 \quad \#\ ,$$

$$\phi_r^d\begin{bmatrix} B_3 & A_2 & , & f_1 & g_1 \\ f_1 & g_1 & & & \end{bmatrix} = B_2 \quad A_1\ ,\ \phi_r^d\begin{bmatrix} \# & A_2 & , & \# & g_2 \\ \# & g_1 & & & \end{bmatrix} = \# \quad A_1\ ,$$

$$\phi_r^d\begin{bmatrix} A_2 & B_3 & , & g_2 & f_2 \\ g_1 & f_1 & & & \end{bmatrix} = A_1 \quad B_2\ ,\ \phi_r^d\begin{bmatrix} B_3 & B_3 & , & f_2 & f_2 \\ f_1 & f_1 & & & \end{bmatrix} = B_2 \quad B_2\ ,$$

$\phi_r^d \begin{bmatrix} A_2 & \# & , & g_2 & \# \\ g_1 & \# & & & \end{bmatrix} = A_1 \quad \# \ , \ \phi_r^d \begin{bmatrix} B_3 & A_2 & , & f_2 & g_2 \\ f_1 & g_1 & & & \end{bmatrix} =$
$B_2 \quad A_1 \Big\}, here \Big\}.$

For $A_2 g_1 \to x A_3 \in P_{21}$ and $B_3 f_1 \to x B_4 \in P_{22}$ we have

$$\text{(DR13)} \quad \Big\{ \Big\{ \phi_r^d \begin{bmatrix} \# & x & , & \# & g_1 \\ \# & A_3 & & & \end{bmatrix} = \begin{matrix} \# & A_2 \\ \# & g_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & x & , & g_1 & f_1 \\ A_3 & B_4 & & & \end{bmatrix} = \begin{matrix} A_2 & B_3 \\ g_1 & f_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & x & , & f_1 & f_1 \\ B_4 & B_4 & & & \end{bmatrix} = \begin{matrix} B_3 & B_3 \\ f_1 & f_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & x & , & f_1 & g_1 \\ B_4 & A_3 & & & \end{bmatrix} = \begin{matrix} B_3 & A_2 \\ f_1 & g_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & \# & , & g_1 & \# \\ A_3 & \# & & & \end{bmatrix} = \begin{matrix} A_2 & \# \\ g_1 & \# \end{matrix}, \ \phi_r^d \begin{bmatrix} \# & x & , & \# & g_2 \\ \# & A_3 & & & \end{bmatrix} = \begin{matrix} \# & A_2 \\ \# & g_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & x & , & g_2 & f_2 \\ A_3 & B_4 & & & \end{bmatrix} = \begin{matrix} A_2 & B_3 \\ g_1 & f_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & x & , & f_2 & f_2 \\ B_4 & B_4 & & & \end{bmatrix} = \begin{matrix} B_3 & B_3 \\ f_1 & f_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & x & , & f_2 & g_2 \\ B_4 & A_3 & & & \end{bmatrix} = \begin{matrix} B_3 & A_2 \\ f_1 & g_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & \# & , & g_2 & \# \\ A_3 & \# & & & \end{bmatrix} =$$
$$\begin{matrix} A_2 & \# \\ g_1 & \# \end{matrix} \Big\}, here \Big\}.$$

For $A_3 g_1 \to x A_3 \in P_{21}$ and $B_4 f_1 \to \bullet B_4 \in P_{22}$ we have

$$\text{(DR14)} \quad \Big\{ \Big\{ \phi_r^d \begin{bmatrix} \# & x & , & \# & g_1 \\ \# & A_3 & & & \end{bmatrix} = \begin{matrix} \# & A_3 \\ \# & g_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & \bullet & , & g_1 & f_1 \\ A_3 & B_4 & & & \end{bmatrix} = \begin{matrix} A_3 & B_4 \\ g_1 & f_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} \bullet & \bullet & , & f_1 & f_1 \\ B_4 & B_4 & & & \end{bmatrix} = \begin{matrix} B_4 & B_4 \\ f_1 & f_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} \bullet & x & , & f_1 & g_1 \\ B_4 & A_3 & & & \end{bmatrix} = \begin{matrix} B_4 & A_3 \\ f_1 & g_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & \# & , & g_1 & \# \\ A_3 & \# & & & \end{bmatrix} = \begin{matrix} A_3 & \# \\ g_1 & \# \end{matrix}, \ \phi_r^d \begin{bmatrix} \# & x & , & \# & g_2 \\ \# & A_3 & & & \end{bmatrix} = \begin{matrix} \# & A_3 \\ \# & g_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & \bullet & , & g_2 & f_2 \\ A_3 & B_4 & & & \end{bmatrix} = \begin{matrix} A_3 & B_4 \\ g_1 & f_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} \bullet & \bullet & , & f_2 & f_2 \\ B_4 & B_4 & & & \end{bmatrix} = \begin{matrix} B_4 & B_4 \\ f_1 & f_1 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} \bullet & x & , & f_2 & g_2 \\ B_4 & A_3 & & & \end{bmatrix} = \begin{matrix} B_4 & A_3 \\ f_1 & g_1 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & \# & , & g_2 & \# \\ A_3 & \# & & & \end{bmatrix} =$$
$$\begin{matrix} A_3 & \# \\ g_1 & \# \end{matrix} \Big\}, here \Big\}.$$

For $A_3 g_2 \to x \in P_{21}$ and $B_4 f_2 \to \bullet \in P_{22}$ we have

$$\text{(DR15)} \quad \Big\{ \Big\{ \phi_r^d \begin{bmatrix} \# & x & , & \lambda & \lambda \\ \# & \# & & & \end{bmatrix} = \begin{matrix} \# & A_3 \\ \# & g_2 \end{matrix}, \ \phi_r^d \begin{bmatrix} x & \bullet & , & \lambda & \lambda \\ \# & \# & & & \end{bmatrix} = \begin{matrix} A_3 & B_4 \\ g_2 & f_2 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} \bullet & \bullet & , & \lambda & \lambda \\ \# & \# & & & \end{bmatrix} = \begin{matrix} B_4 & B_4 \\ f_2 & f_2 \end{matrix}, \ \phi_r^d \begin{bmatrix} \bullet & x & , & \lambda & \lambda \\ \# & \# & & & \end{bmatrix} = \begin{matrix} B_4 & A_3 \\ f_2 & g_2 \end{matrix},$$
$$\phi_r^d \begin{bmatrix} x & \# & , & \lambda & \lambda \\ \# & \# & & & \end{bmatrix} = \begin{matrix} A_3 & \# \\ g_2 & \# \end{matrix} \Big\}, here \Big\}.$$

To delete the border symbols $\#'s$ we have the following deletion rules:

$$\text{(DR16)}\quad \left\{\left\{\varphi_r^d\left[\# \;\; x \;\; , \;\; \lambda \;\; \lambda\right] = \# \;\; \# \;, \; \varphi_r^d\left[x \;\; \bullet \;\; , \;\; \lambda \;\; \lambda\right] = \# \;\; \# \;,\right.\right.$$

$$\varphi_r^d\left[\bullet \;\; \bullet \;\; , \;\; \lambda \;\; \lambda\right] = \# \;\; \# \;, \; \varphi_r^d\left[\bullet \;\; x \;\; , \;\; \lambda \;\; \lambda\right] = \# \;\; \# \;,$$

$$\varphi_r^d\left[x \;\; \# \;\; , \;\; \lambda \;\; \lambda\right] = \# \;\; \# \;, \; \varphi_r^d\left[\lambda \;\; \lambda \;\; , \;\; \# \;\; x\right] = \# \;\; \# \;,$$

$$\varphi_r^d\left[\lambda \;\; \lambda \;\; , \;\; x \;\; \bullet\right] = \# \;\; \# \;, \; \varphi_r^d\left[\lambda \;\; \lambda \;\; , \;\; \bullet \;\; \bullet\right] = \# \;\; \# \;,$$

$$\varphi_r^d\left[\lambda \;\; \lambda \;\; , \;\; \bullet \;\; x\right] = \# \;\; \# \;, \; \varphi_r^d\left[\lambda \;\; \lambda \;\; , \;\; x \;\; \#\right] = \# \;\; \# \;,$$

$$\left.\left.\varphi_c^d\begin{bmatrix} \lambda & x \\ \lambda \,, & x \end{bmatrix} = \begin{matrix} \# \\ \# \end{matrix} \;, \; \varphi_c^d\begin{bmatrix} x & \lambda \\ x \,, & \lambda \end{bmatrix} = \begin{matrix} \# \\ \# \end{matrix}\right\} \;, \; here\right\}.$$

Thus we have constructed rules of the P system corresponding to the rules of the grammar G.

A sample generation of tokens of H of $\boldsymbol{x}$'s with exactly horizontal row of $\boldsymbol{x}$'s in the middle using the above rules is given below:

(blue) - is inserted context

(red) - is to be deleted context

$$\begin{matrix} \# & \# & \# \\ \# & S & \# \end{matrix} \Rightarrow^{col_i}_{2,IR1} \begin{matrix} \# & \color{blue}{\#} & \color{blue}{\#} & \# & \# \\ \# & \color{blue}{S_1} & \color{blue}{A} & S & \# \end{matrix} = \begin{matrix} \# & \# & \# & \color{red}{\#} & \# \\ \# & S_1 & A & \color{red}{S} & \# \end{matrix} \Rightarrow^{col_d}_{2,DR1}$$

$$\begin{matrix} \# & \# & \# & \# \\ \# & S_1 & A & \# \end{matrix} \Rightarrow^{col_i}_{2,IR2} \begin{matrix} \# & \# & \color{blue}{\#} & \color{blue}{\#} & \# & \# \\ \# & S_1 & \color{blue}{S_2} & \color{blue}{A} & A & \# \end{matrix} = \begin{matrix} \# & \# & \# & \# & \color{red}{\#} & \# \\ \# & S_1 & S_2 & A & \color{red}{A} & \# \end{matrix} \Rightarrow^{col_d}_{2,DR2}$$

$$\begin{matrix} \# & \# & \# & \# & \# \\ \# & S_1 & S_2 & A & \# \end{matrix} \Rightarrow^{col_i}_{2,IR2} \begin{matrix} \# & \# & \# & \color{blue}{\#} & \color{blue}{\#} & \# & \# \\ \# & S_1 & S_2 & \color{blue}{S_2} & \color{blue}{A} & A & \# \end{matrix} =$$

$$\begin{matrix} \# & \# & \# & \# & \# & \color{red}{\#} & \# \\ \# & S_1 & S_2 & S_2 & A & \color{red}{A} & \# \end{matrix} \Rightarrow^{col_d}_{2,DR2} \begin{matrix} \# & \# & \# & \# & \# & \# \\ \# & S_1 & S_2 & S_2 & A & \# \end{matrix} \Rightarrow^{col_i}_{2,IR3}$$

$$\begin{matrix} \# & \# & \# & \# & \color{blue}{\#} & \color{blue}{\#} & \# & \# \\ \# & S_1 & S_2 & S_2 & \color{blue}{S_2} & \color{blue}{S_1} & A & \# \end{matrix} = \begin{matrix} \# & \# & \# & \# & \# & \# & \color{red}{\#} & \# \\ \# & S_1 & S_2 & S_2 & S_2 & S_1 & \color{red}{A} & \# \end{matrix} \Rightarrow^{col_d}_{2,DR3}$$

$$\begin{matrix} \# & \# & \# & \# & \# & \# & \# \\ \# & S_1 & S_2 & S_2 & S_2 & S_1 & \# \end{matrix} \Rightarrow^{row_i}_{2,IR4} \begin{matrix} \# & \# & \# & \# & \# & \# & \# \\ \color{blue}{\#} & \color{blue}{x} & \color{blue}{\bullet} & \color{blue}{\bullet} & \color{blue}{\bullet} & \color{blue}{x} & \color{blue}{\#} \\ \color{blue}{\#} & \color{blue}{A_1} & \color{blue}{B_1} & \color{blue}{B_1} & \color{blue}{B_1} & \color{blue}{A_1} & \color{blue}{\#} \\ \color{blue}{\#} & \color{blue}{g_2} & \color{blue}{f_2} & \color{blue}{f_2} & \color{blue}{f_2} & \color{blue}{g_2} & \color{blue}{\#} \\ \# & S_1 & S_2 & S_2 & S_2 & S_1 & \# \end{matrix} =$$

$$\begin{matrix} \# & \# & \# & \# & \# & \# & \# \\ \# & x & \bullet & \bullet & \bullet & x & \# \\ \# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\ \# & g_2 & f_2 & f_2 & f_2 & g_2 & \# \\ \color{red}{\#} & \color{red}{S_1} & \color{red}{S_2} & \color{red}{S_2} & \color{red}{S_2} & \color{red}{S_1} & \color{red}{\#} \end{matrix} \Rightarrow^{row_d}_{2,DR4} \begin{matrix} \# & \# & \# & \# & \# & \# & \# \\ \# & x & \bullet & \bullet & \bullet & x & \# \\ \# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\ \# & g_2 & f_2 & f_2 & f_2 & g_2 & \# \end{matrix} \Rightarrow^{row_i}_{2,IR6}$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
=
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR6}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR7}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_2 & B_2 & B_2 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
=
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_2 & B_2 & B_2 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_1 & B_1 & B_1 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR6}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_1 & B_2 & B_2 & B_2 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR8}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_2 & B_3 & B_3 & B_3 & A_2 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_2 & B_2 & B_2 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
=
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_2 & B_3 & B_3 & B_3 & A_2 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & A_1 & B_2 & B_2 & B_2 & A_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR12}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_2 & B_3 & B_3 & B_3 & A_2 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR9}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & A_2 & B_3 & B_3 & B_3 & A_2 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array} =
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & A_2 & B_3 & B_3 & B_3 & A_2 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR13}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR10}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array} =
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR14}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR10}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_1 & f_1 & f_1 & f_1 & g_1 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array} =
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\color{red}\# & \color{red}A_3 & \color{red}B_4 & \color{red}B_4 & \color{red}B_4 & \color{red}A_3 & \color{red}\# \\
\color{red}\# & \color{red}g_1 & \color{red}f_1 & \color{red}f_1 & \color{red}f_1 & \color{red}g_1 & \color{red}\# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_d}_{2,DR14}
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
\Rightarrow^{row_i}_{2,IR11}
$$

$$
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{blue}\# & \color{blue}x & \color{blue}\bullet & \color{blue}\bullet & \color{blue}\bullet & \color{blue}x & \color{blue}\# \\
\color{blue}\# & \color{blue}\# & \color{blue}\# & \color{blue}\# & \color{blue}\# & \color{blue}\# & \color{blue}\# \\
\# & A_3 & B_4 & B_4 & B_4 & A_3 & \# \\
\# & g_2 & f_2 & f_2 & f_2 & g_2 & \#
\end{array}
=
\begin{array}{ccccccc}
\# & \# & \# & \# & \# & \# & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & \# & \# & \# & \# & \# & \# \\
\color{red}\# & \color{red}A_3 & \color{red}B_4 & \color{red}B_4 & \color{red}B_4 & \color{red}A_3 & \color{red}\# \\
\color{red}\# & \color{red}g_2 & \color{red}f_2 & \color{red}f_2 & \color{red}f_2 & \color{red}g_2 & \color{red}\#
\end{array}
\Rightarrow^{row_d}_{2,DR15}
$$

$$
\begin{array}{ccccccc}
\color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & \# & \# & \# & \# & \# & \#
\end{array}
\Rightarrow^{row_d}_{2,DR16}
\begin{array}{ccccccc}
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & x & x & x & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\# & \color{red}\#
\end{array}
\Rightarrow^{row_d}_{2,DR16}
$$

$$
\begin{array}{ccccccc}
\color{red}\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & x & x & x & x & x & \# \\
\color{red}\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & x & \bullet & \bullet & \bullet & x & \# \\
\color{red}\# & x & \bullet & \bullet & \bullet & x & \#
\end{array}
\Rightarrow^{col_d}_{2,DR16}
\begin{array}{cccccc}
x & \bullet & \bullet & \bullet & x & \color{red}\# \\
x & \bullet & \bullet & \bullet & x & \color{red}\# \\
x & \bullet & \bullet & \bullet & x & \color{red}\# \\
x & x & x & x & x & \color{red}\# \\
x & \bullet & \bullet & \bullet & x & \color{red}\# \\
x & \bullet & \bullet & \bullet & x & \color{red}\# \\
x & \bullet & \bullet & \bullet & x & \color{red}\#
\end{array}
\Rightarrow^{col_d}_{2,DR16}
\begin{array}{ccccc}
x & \bullet & \bullet & \bullet & x \\
x & \bullet & \bullet & \bullet & x \\
x & \bullet & \bullet & \bullet & x \\
x & x & x & x & x \\
x & \bullet & \bullet & \bullet & x \\
x & \bullet & \bullet & \bullet & x \\
x & \bullet & \bullet & \bullet & x
\end{array}
.
$$