Symmetries of $\mathbb{Z}_N$ graded discrete integrable systems

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March 24, 2020

Abstract

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems (lattice equations). We discuss differential-difference equations which then we interpret as symmetries of the discrete systems. In particular, we present nonlocal symmetries which are associated with the $2D$ Toda lattice.

Keywords: Discrete integrable system, Lax pair, symmetry, Toda lattice.
MSC: 37K05, 37K10, 37K35, 39A05

1 Introduction

We recently introduced a class of $\mathbb{Z}_N$ graded discrete Lax pairs and studied the associated discrete integrable systems [7]. Many well known examples belong to that scheme for $N = 2$, so, for $N \geq 3$, some of our systems may be regarded as generalisations of these. The purpose of this paper is to consider *continuous deformations* of our discrete Lax matrices, giving rise to differential-difference equations, which can be interpreted as continuous (local) symmetries of our fully discrete systems. Again, the $\mathbb{Z}_N$ grading gives us a way of systematically calculating general formulae for our generic systems. We must consider various types of compatibility. For partial differential equations, compatibility is just the usual integrability conditions (equality of mixed partial derivatives). Our partial difference equations require a similar compatibility of *mixed shifts*, whilst our differential-difference equations require a compatibility between *shifts* and a *continuous* variable.

In Section 2 we give a short review of the general framework for the derivation of discrete systems compatible with our discrete Lax pairs, given in [7]. In particular, we introduce a pair of *potential forms* of our equations. In Section 3 we consider the continuous isospectral flows of our Lax matrix L and give the explicit form of the first flow for the *general* case. We also derive the form of the master symmetry, which can be used to construct a hierarchy of isospectral flows. In Section 4 we consider the compatibility of these isospectral flows with the Lax pair L and M for the fully discrete system, thus giving them the role of *symmetries*. These symmetries are also written in terms of the potential variables.

In Section 5 we consider the class of degenerate systems, the simplest example being Hirota's KdV equation. We present a generalisation of Hirota's KdV equation, giving a scalar equation, defined on $2N$ points. We show that one of its symmetries is Miura related to a Bogoyavlenskii lattice equation.

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In Section 6 we present two *nonlocal symmetries* for our general discrete system. These are associated with forms of the $2D$ Toda lattice [9, 6], when we use the potential forms. In particular, in the quotient potential form, the nonlocal symmetries act as Bäcklund transformations for this Toda lattice. In the generic (non-reduced) case, our fully discrete system is just the corresponding nonlinear superposition principle. This connection with the Toda lattice was to be expected, since this nonlinear superposition formula contains, as special cases, several well known examples of discrete integrable system, including the modified KdV and modified Boussinesq equations (see [11]). The connection of the $2D$ Toda lattice to these "modified" (PDE) hierarchies was given in [6] in the context of the factorisation of scalar Lax operators [4, 5] and the whole hierarchy shares the same Bianchi superposition formula.

2 $\mathbb{Z}_N$-Graded Lax Pairs

We now consider the specific discrete Lax pairs, which we introduced in [7]. Consider a pair of matrix equations of the form

$$\Psi_{m+1,n} = L_{m,n}\,\Psi_{m,n} \equiv \left(U_{m,n} + \lambda\,\Omega^{\ell_1}\right)\,\Psi_{m,n}, \tag{2.1a}$$

$$\Psi_{m,n+1} = M_{m,n}\,\Psi_{m,n} \equiv \left(V_{m,n} + \lambda\,\Omega^{\ell_2}\right)\,\Psi_{m,n}, \tag{2.1b}$$

where

$$U_{m,n} = \operatorname{diag}\left(u_{m,n}^{(0)},\dots,u_{m,n}^{(N-1)}\right)\Omega^{k_1}, \quad V_{m,n} = \operatorname{diag}\left(v_{m,n}^{(0)},\dots,v_{m,n}^{(N-1)}\right)\Omega^{k_2}, \tag{2.1c}$$

and Ω is an $N\times N$ matrix, defined by

$$(\Omega)_{i,j} = \delta_{j-i,1} + \delta_{i-j,N-1}, \quad \text{satisfying} \quad \Omega^N = I_N.$$

The matrix Ω defines a grading, which we call the *level*:

Definition 2.1 (A level k matrix) *An $N\times N$ matrix A of the form*

$$A = \operatorname{diag}\left(a^{(0)},\dots,a^{(N-1)}\right)\Omega^k$$

will be said to have level k, written $lev(A) = k$.

The four matrices of (2.1) are then seen to be of respective levels k_i, ℓ_i, with $\ell_i \neq k_i$ (for each i). The Lax pair is characterised by the quadruple $(k_1,\ell_1;k_2,\ell_2)$, which we refer to as *the level structure* of the system, and for consistency, we require

$$k_1 + \ell_2 \equiv k_2 + \ell_1 \ (\mathrm{mod} N). \tag{2.2}$$

Since matrices U, V and Ω are independent of λ, the compatibility condition of (2.1),

$$L_{m,n+1}M_{m,n} = M_{m+1,n}L_{m,n}, \tag{2.3}$$

splits into the system

$$U_{m,n+1}V_{m,n} = V_{m+1,n}U_{m,n}\,, \tag{2.4a}$$

$$U_{m,n+1}\Omega^{\ell_2} - \Omega^{\ell_2}U_{m,n} = V_{m+1,n}\Omega^{\ell_1} - \Omega^{\ell_1}V_{m,n}, \tag{2.4b}$$

which can be written explicitly as

$$u_{m,n+1}^{(i)}v_{m,n}^{(i+k_1)} = v_{m+1,n}^{(i)}u_{m,n}^{(i+k_2)}\,, \tag{2.5a}$$

$$u_{m,n+1}^{(i)} - u_{m,n}^{(i+\ell_2)} = v_{m+1,n}^{(i)} - v_{m,n}^{(i+\ell_1)}\,, \tag{2.5b}$$

for $i \in \mathbb{Z}_N$.

Notation: The symbols $\mathcal{S}_m$ and $\mathcal{S}_n$ will respectively denote the shifts in the m and n directions: $\mathcal{S}_m u^{(i)}_{m,n} = u^{(i)}_{m+1,n}$ and $\mathcal{S}_n u^{(i)}_{m,n} = u^{(i)}_{m,n+1}$, with $\Delta_m = \mathcal{S}_m - 1$ and $\Delta_n = \mathcal{S}_n - 1$ the corresponding *differences*.

2.1 Classification Problem

In [7] we discussed the equivalence problem of two systems with level structures $(k_1, \ell_1; k_2, \ell_2)$ and $(k_1', \ell_1'; k_2', \ell_2')$, satisfying (2.2). They were said to be *equivalent*, if one quadruple can be mapped to the other by applying either of the following transformations.

$$\begin{aligned} \mathcal{T}_1 &: \quad (a,b;c,d) \mapsto (c,d;a,b) \\ \mathcal{T}_2 &: \quad (a,b;c,d) \mapsto (N-a, N-b; N-c, N-d)\,. \end{aligned} \tag{2.6}$$

We can then classify our Lax pairs, depending on whether or not N and $\ell_i - k_i$, $i = 1, 2$, are coprime.

1. *The coprime case:* $\big(N, \ell_1 - k_1\big) = \big(N, \ell_2 - k_2\big) = 1$.
   This involves Lax pairs which satisfy

   $$\prod_{j=0}^{N-1} u^{(j)}_{m,n} \;=\; a, \qquad \prod_{j=0}^{N-1} v^{(j)}_{m,n} \;=\; b, \quad \text{where} \quad \Delta_n a = \Delta_m b = 0. \tag{2.7}$$

   The above relations allow us to express one function from each set in terms of the remaining ones. The coprime case is further subdivided into:

   - *The generic subcase :* $ab \neq 0$,
   - *The degenerate subcase :* $a \neq 0$, $b = 0$.

   Lax pairs with $a = 0$, $b \neq 0$ are equivalent to the above degenerate case by a change of independent variables. Finally, the fully degenerate case $a = b = 0$ is empty.

2. *The non-coprime case:* $\big(N, \ell_1 - k_1\big) = \big(N, \ell_2 - k_2\big) = p > 1$
   If $N = pq$, then the Lax pairs take the form of $p \times p$ matrices of $q \times q$ blocks.

   The fundamental variables now form $q \times q$ blocks, which have various degrees of coupling/decoupling, depending upon the values of k_1 and k_2.

2.2 Potential Forms

We can reduce equations (2.5) by introducing potentials.

2.2.1 The Quotient Potential

Equations (2.5a) hold identically if we set

$$u^{(i)}_{m,n} \;=\; \alpha\, \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}}\,, \quad v^{(i)}_{m,n} \;=\; \beta\, \frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}}\,, \quad i \in \mathbb{Z}_N, \tag{2.8a}$$

where $a = \alpha^N$, $b = \beta^N$, after which (2.5b) takes the form

$$\alpha \left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_1)}_{m,n+1}} - \frac{\phi^{(i+\ell_2)}_{m+1,n}}{\phi^{(i+\ell_2+k_1)}_{m,n}} \right) \;=\; \beta \left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_2)}_{m+1,n}} - \frac{\phi^{(i+\ell_1)}_{m,n+1}}{\phi^{(i+\ell_1+k_2)}_{m,n}} \right)\,, \quad i \in \mathbb{Z}_N, \tag{2.8b}$$

defined on a square lattice. These equations can be explicitly solved for the variables on any of the four vertices and, in particular,

$$\phi^{(i)}_{m+1,n+1} \;=\; \frac{\phi^{(i+k_1)}_{m,n+1}\phi^{(i+k_2)}_{m+1,n}}{\phi^{(i+k_1+\ell_2)}_{m,n}}\;\left(\frac{\alpha\,\phi^{(i+\ell_2)}_{m+1,n}-\beta\,\phi^{(i+\ell_1)}_{m,n+1}}{\alpha\,\phi^{(i+k_2)}_{m+1,n}-\beta\,\phi^{(i+k_1)}_{m,n+1}}\right)\,,\quad i\in\mathbb{Z}_N. \tag{2.8c}$$

In this potential form, the Lax pair (2.1) can be written

$$\begin{aligned}\Psi_{m+1,n} \;&=\; \left(\alpha\boldsymbol{\phi}_{m+1,n}\Omega^{k_1}\boldsymbol{\phi}^{-1}_{m,n}+\lambda\Omega^{\ell_1}\right)\Psi_{m,n},\\ \Psi_{m,n+1} \;&=\; \left(\beta\boldsymbol{\phi}_{m,n+1}\Omega^{k_2}\boldsymbol{\phi}^{-1}_{m,n}+\lambda\Omega^{\ell_2}\right)\Psi_{m,n},\end{aligned} \tag{2.9a}$$

where

$$\boldsymbol{\phi}_{m,n} \;:=\; \mathrm{diag}\left(\phi^{(0)}_{m,n},\dots,\phi^{(N-1)}_{m,n}\right)\quad\text{and}\quad \det\left(\boldsymbol{\phi}_{m,n}\right)=\prod_{i=0}^{N-1}\phi^{(i)}_{m,n}=1. \tag{2.9b}$$

We can then show that the Lax pair (2.9) is compatible if and only if the system (2.8b) holds.

2.2.2 The Additive Potential

Equations (2.5b) hold identically if we set

$$u^{(i)}_{m,n}=\chi^{(i)}_{m+1,n}-\chi^{(i+\ell_1)}_{m,n},\quad v^{(i)}_{m,n}=\chi^{(i)}_{m,n+1}-\chi^{(i+\ell_2)}_{m,n},\quad i\in\mathbb{Z}_N. \tag{2.10a}$$

Equations (2.5a) then take the form

$$\frac{\left(\chi^{(i)}_{m+1,n+1}-\chi^{(i+\ell_1)}_{m,n+1}\right)}{\left(\chi^{(i+k_2)}_{m+1,n}-\chi^{(i+k_2+\ell_1)}_{m,n}\right)}=\frac{\left(\chi^{(i)}_{m+1,n+1}-\chi^{(i+\ell_2)}_{m+1,n}\right)}{\left(\chi^{(i+k_1)}_{m,n+1}-\chi^{(i+k_1+\ell_2)}_{m,n}\right)}, \tag{2.10b}$$

for $i\in\mathbb{Z}_N$.

In this potential form, the Lax pair (2.1) can be written

$$\begin{aligned}\Psi_{m+1,n} \;&=\; \Big(\left(\boldsymbol{\chi}_{m+1,n}-\Omega^{\ell_1}\boldsymbol{\chi}_{m,n}\Omega^{-\ell_1}\right)\Omega^{k_1}+\lambda\Omega^{\ell_1}\Big)\Psi_{m,n},\\ \Psi_{m,n+1} \;&=\; \Big(\left(\boldsymbol{\chi}_{m,n+1}-\Omega^{\ell_2}\boldsymbol{\chi}_{m,n}\Omega^{-\ell_2}\right)\Omega^{k_2}+\lambda\Omega^{\ell_2}\Big)\Psi_{m,n},\end{aligned} \tag{2.11}$$

where

$$\boldsymbol{\chi}_{m,n} \;:=\; \mathrm{diag}\left(\chi^{(0)}_{m,n},\cdots,\chi^{(N-1)}_{m,n}\right).$$

We can then show that the Lax pair (2.11) is compatible if and only if the system (2.10b) holds.

In this case, conditions (2.7) become the first integrals

$$\prod_{i=0}^{N-1}\left(\chi^{(i)}_{m+1,n}-\chi^{(i+\ell_1)}_{m,n}\right)\;=\;\alpha^N,\quad \prod_{i=0}^{N-1}\left(\chi^{(i)}_{m,n+1}-\chi^{(i+\ell_2)}_{m,n}\right)\;=\;\beta^N, \tag{2.12}$$

where we have set $a=\alpha^N$, $b=\beta^N$. Hence it is not always possible to reduce the number of potentials χ (in *local* terms) by employing these.

3 Differential-Difference Equations as Isospectral Flows

In this section we consider semi-discrete Lax pairs, involving both discrete and continuous variables; the resulting isospectral flows are differential-difference equations. We first discuss continuous isospectral deformations of the matrix $L_{m,n}$, satisfying (2.1a). The simplest solutions give rise to the lowest order autonomous flow $\partial_{t^1} u^{(i)}_{m,n}$ and a *master symmetry*, which generates an infinite hierarchy of autonomous flows.

In this section we consider differential-difference systems which only involve shifts in the discrete variable m. However, since we later wish to interpret these as symmetries of the fully-discrete system (2.5), we continue to write our functions as depending on *both* indices m and n. We have also deliberately not labelled (k, ℓ) as (k_1, ℓ_1), since when we later replace $L_{m,n}$ by $M_{m,n}$, we will need $(k, \ell) = (k_2, \ell_2)$.

Let us start with the $N \times N$ Lax pair

$$\Psi_{m+1,n} \;=\; L_{m,n}\Psi_{m,n}\,, \quad \partial_t \Psi_{m,n} \;=\; S_{m,n}\,\Psi_{m,n}\,, \tag{3.1}$$

with $L_{m,n}$ defined by (2.1a) (but with the replacement $(k_1, \ell_1) \to (k, \ell)$) and $S_{m,n}$ a $\lambda-$dependent matrix, to be determined.

The compatibility condition of the above system is

$$\partial_t L_{m,n} = S_{m+1,n}L_{m,n} - L_{m,n}S_{m,n}. \tag{3.2}$$

If we write this as

$$L^{-1}_{m,n}\partial_t L_{m,n} = L^{-1}_{m,n}S_{m+1,n}L_{m,n} - S_{m,n},$$

and note that

$$\partial_t \log\left(\det L_{m,n}\right) = \mathrm{Tr}\left(L^{-1}_{m,n}\partial_t L_{m,n}\right),$$

we obtain the conservation law

$$\partial_t \log\left(\det L_{m,n}\right) = \Delta_m\ \left(\mathrm{Tr}\left(S_{m,n}\right)\right). \tag{3.3}$$

Setting

$$S_{m,n} = L^{-1}_{m,n}Q_{m,n}, \tag{3.4}$$

we can then write (3.2) as

$$\left(U_{m+1,n} + \lambda\Omega^{\ell}\right)\left(\partial_t U_{m,n} + Q_{m,n}\right) = Q_{m+1,n}\left(U_{m,n} + \lambda\Omega^{\ell}\right), \tag{3.5}$$

which is used to determine $Q_{m,n}$ in terms of $U_{m,n}$, as well as the resulting differential-difference equations.

For the simplest solution, we assume that $Q_{m,n}$ is independent of the spectral parameter. We then collect the different powers of λ in equation (3.5):

$$Q_{m,n}U_{m-1,n}\ -\ U_{m,n}\Omega^{-\ell}Q_{m,n}\Omega^{\ell} = 0, \tag{3.6a}$$

$$\partial_t U_{m,n} = \Omega^{-\ell}Q_{m+1,n}\Omega^{\ell}\ -\ Q_{m,n}. \tag{3.6b}$$

Clearly, the second equation implies that $lev(Q_{m,n}) = lev(U_{m,n})$ and provides us with a set of N differential-difference equations for the functions $u^{(i)}$, $i \in \mathbb{Z}_N$.

We write

$$U_{m,n} = \mathrm{diag}(u^{(0)}_{m,n}, u^{(1)}_{m,n}, \ldots, u^{(N-1)}_{m,n})\Omega^k, \quad Q_{m,n} = \mathrm{diag}(q^{(0)}_{m,n}, q^{(1)}_{m,n}, \ldots, q^{(N-1)}_{m,n})\Omega^k. \tag{3.7}$$

Then, equations (3.6) give

$$q_{m,n}^{(i)}\, u_{m-1,n}^{(i+k)} = u_{m,n}^{(i)}\, q_{m,n}^{(i+k-\ell)}, \tag{3.8a}$$

$$\partial_t u_{m,n}^{(i)} = q_{m+1,n}^{(i-\ell)} - q_{m,n}^{(i)}. \tag{3.8b}$$

Remark 3.1 *Notice that (3.8a) implies that*

$$\prod_{i=0}^{N-1} \frac{q_{m,n}^{(i)}}{q_{m,n}^{(i+k-\ell)}} = \prod_{i=0}^{N-1} \frac{u_{m,n}^{(i)}}{u_{m-1,n}^{(i+k)}}.$$

The left hand side equals 1, which tells us that $\prod_{i=0}^{N-1} u_{m,n}^{(i)}$ is a constant (in m) (compare with (2.7), which tells us that it is independent of n).

The existence of this isospectral flow therefore implies that a and b of (2.7) are independent of both m and n.

To calculate $(\mathrm{Tr}(L_{m,n}^{-1}\, Q_{m,n})$, we note that

$$L_{m,n}^{-1} = \Omega^{-\ell}\left(\lambda I_N + U\Omega^{-\ell}\right)^{-1} = \frac{1}{\lambda^N - (-1)^N a}\, \Omega^{-\ell}\left(\lambda^{N-1} I_N - \lambda^{N-2}\mathcal{D} + \cdots + (-\mathcal{D})^{N-1}\right),$$

where $\mathcal{D} = U\Omega^{-\ell}$. This follows from the property $\mathcal{D}^N = a\, I_N$ in the coprime case. Thus

$$\mathrm{Tr}\left(L_{m,n}^{-1} Q_{m,n}\right) = \frac{1}{\lambda^N - (-1)^N a}\, \mathrm{Tr}\left(\Omega^{-\ell}\left(\lambda^{N-1} I_N - \lambda^{N-2}\mathcal{D} + \cdots + (-\mathcal{D})^{N-1}\right) Q_{m,n}\right).$$

Since $(N, k-\ell) = 1$, the only diagonal term (level $N(k-\ell)$) is $\Omega^{-\ell}(-\mathcal{D})^{N-1} Q_{m,n}$, and $\mathrm{Tr}\left(\Omega^{-\ell}(-\mathcal{D})^{N-1} Q_{m,n}\right) = -\,\mathrm{Tr}\left((-\mathcal{D})^N U_{m,n}^{-1} Q_{m,n}\right)$, so we have

$$\mathrm{Tr}\left(L_{m,n}^{-1} Q_{m,n}\right) = \frac{a}{a - (-\lambda)^N} \sum_{i=0}^{N-1} \frac{q_{m,n}^{(i)}}{u_{m,n}^{(i)}}.$$

Noting that both a and λ are independent of m and that $\det(L_{m,n}) = a - (-\lambda)^N$, the conservation law (3.3) implies

$$\partial_t (a - (-\lambda)^N) = a\, \Delta_m \left(\sum_{i=0}^{N-1} \frac{q_{m,n}^{(i)}}{u_{m,n}^{(i)}}\right). \tag{3.9}$$

Since the left hand side of this is independent of m, we have

$$\sum_{i=0}^{N-1} \frac{q_{m,n}^{(i)}}{u_{m,n}^{(i)}} = \frac{c_0 + c_1 m}{a} \quad \text{and} \quad \partial_t a = c_1, \tag{3.10}$$

where c_0, c_1 are constants.

Equations (3.8a) and (3.10), fully determine the functions $q_{m,n}^{(i)}$ in terms of $\mathbf{u}_{m,n}$ and $\mathbf{u}_{m-1,n}$.

Remark 3.2 (Explicit formula for $q_{m,n}^{(i)}$) *It is, in fact, possible to derive a general formula for the functions $q_{m,n}^{(i)}$:*

$$q_{m,n}^{(i\delta)} = q_{m,n}^{(0)} \prod_{j=0}^{i-1} \frac{u_{m-1,n}^{(j\delta+k)}}{u_{m,n}^{(j\delta)}}, \quad \textit{where}\ \ \delta = k - \ell\,,\ \ i = 1, \ldots, N-1, \tag{3.11}$$

and where $q^{(0)}_{m,n}$ *is given by*

$$\sum_{i=0}^{N-1} \frac{q^{(i)}_{m,n}}{u^{(i)}_{m,n}} = \frac{1}{a} \quad \Rightarrow \quad q^{(0)}_{m,n} = \frac{u^{(0)}_{m,n}}{a}\left(1+\sum_{i=1}^{N-1}\frac{u^{(0)}_{m,n}}{u^{(i\delta)}_{m,n}}\prod_{j=0}^{i-1}\frac{u^{(j\delta+k)}_{m-1,n}}{u^{(j\delta)}_{m,n}}\right)^{-1}.$$

Thus, we have proven

Proposition 3.3 *The system*

$$\Psi_{m+1,n} = (U_{m,n}+\lambda\Omega^{\ell})\Psi_{m,n}, \quad \partial_t\Psi_{m,n} = (U_{m,n}+\lambda\Omega^{\ell})^{-1} Q_{m,n}\,\Psi_{m,n},$$

where

$$U_{m,n} = \mathrm{diag}(u^{(0)}_{m,n}, u^{(1)}_{m,n}, \ldots, u^{(N-1)}_{m,n})\Omega^k, \quad Q_{m,n} = \mathrm{diag}(q^{(0)}_{m,n}, q^{(1)}_{m,n}, \ldots, q^{(N-1)}_{m,n})\Omega^k,$$

with

$$q^{(i)}_{m,n}\, u^{(i+k)}_{m-1,n} = u^{(i)}_{m,n}\, q^{(i+k-\ell)}_{m,n} \quad \textit{and} \quad \sum_{i=0}^{N-1}\frac{q^{(i)}_{m,n}}{u^{(i)}_{m,n}} = \frac{1}{a},$$

is compatible if and only if $u^{(i)}_{m,n}$ *satisfies*

$$\partial_t u^{(i)}_{m,n} = q^{(i-\ell)}_{m+1,n} - q^{(i)}_{m,n}. \tag{3.12}$$

This leads to the (differential-difference) local conservation law

$$\partial_t\left(\sum_{i=0}^{N-1} u^{(i)}_{m,n}\right) = \Delta_m\left(\sum_{i=0}^{N-1} q^{(i)}_{m,n}\right). \tag{3.13}$$

An equivalent statement can be made for the case $c_0 = 0, c_1 = 1$*, corresponding to the flow*

$$\partial_\tau u^{(i)}_{m,n} = (m+1)q^{(i-\ell)}_{m+1,n} - mq^{(i)}_{m,n}, \quad \textit{with} \quad \partial_\tau a = 1, \tag{3.14}$$

by making the replacement $q^{(i)}_{m,n} \to mq^{(i)}_{m,n}$ *in the formulae.*

We will see in Section 3.2 that this defines a *master symmetry* for a hierarchy of autonomous flows, the first of which is (3.12).

3.1 All Inequivalent Cases for $N = 2$ and $N = 3$

Up to equivalence defined by

$$\mathcal{T} : (a, b) \mapsto (N-a, N-b),$$

we list all semi-discrete Lax pairs in two and three dimensions. In the following lists, we present only the entries of matrices Q, for the autonomous case, and the corresponding differential-difference equations. To obtain the non-autonomous solution, with $c_0 = 0, c_1 = 1$, we just multiply the formula for $q^{(i)}_{m,n}$ by m, obtaining the $\tau-$flow (3.14).

3.1.1 The Case $N = 2$

Here there are just two cases.

1. **Level structure** $(k,\ell) = (0,1)$. Entries of matrix $Q_{m,n}$ are

$$q^{(0)}_{m,n} \;=\; \frac{1}{u^{(0)}_{m-1,n} + u^{(1)}_{m,n}}\,, \quad q^{(1)}_{m,n} \;=\; \frac{u^{(0)}_{m-1,n}}{u^{(0)}_{m,n}\left(u^{(0)}_{m-1,n} + u^{(1)}_{m,n}\right)}.$$

The corresponding differential-difference equations are

$$\partial_t u^{(0)}_{m,n} \;=\; q^{(1)}_{m+1,n} - q^{(0)}_{m,n}\,, \quad \partial_t u^{(1)}_{m,n} \;=\; q^{(0)}_{m+1,n} - q^{(1)}_{m,n}\,. \tag{3.15}$$

2. **Level structure** $(k,\ell) = (1,0)$. Entries of matrix $Q_{m,n}$ are

$$q^{(0)}_{m,n} \;=\; \frac{1}{u^{(1)}_{m-1,n} + u^{(1)}_{m,n}}\,, \quad q^{(1)}_{m,n} \;=\; \frac{u^{(1)}_{m-1,n}}{u^{(0)}_{m,n}\left(u^{(1)}_{m-1,n} + u^{(1)}_{m,n}\right)}. \tag{3.16a}$$

The corresponding differential-difference equations are

$$\partial_t u^{(0)}_{m,n} \;=\; q^{(0)}_{m+1,n} - q^{(0)}_{m,n}\,, \quad \partial_t u^{(1)}_{m,n} \;=\; q^{(1)}_{m+1,n} - q^{(1)}_{m,n}\,. \tag{3.16b}$$

3.1.2 The Case $N = 3$

Here there are three cases.

1. **Level structure** $(k,\ell) = (0,1)$. Entries of matrix $Q_{m,n}$ are

$$q^{(0)}_{m,n} = \frac{u^{(1)}_{m-1,n}}{u^{(1)}_{m,n}}\, q^{(1)}_{m,n}, \qquad q^{(1)}_{m,n} = \frac{1}{\Gamma}, \qquad q^{(2)}_{m,n} = \frac{u^{(0)}_{m-1,n}u^{(1)}_{m-1,n}}{u^{(0)}_{m,n}u^{(1)}_{m,n}}\, q^{(1)}_{m,n}, \tag{3.17a}$$

where $\Gamma = u^{(0)}_{m-1,n}u^{(1)}_{m-1,n} + u^{(0)}_{m,n}u^{(2)}_{m,n} + u^{(1)}_{m-1,n}u^{(2)}_{m,n}$. The corresponding differential-difference equations are

$$\partial_t u^{(0)}_{m,n} \;=\; q^{(2)}_{m+1,n} - q^{(0)}_{m,n}\,, \quad \partial_t u^{(1)}_{m,n} \;=\; q^{(0)}_{m+1,n} - q^{(1)}_{m,n}\,, \quad \partial_t u^{(2)}_{m,n} \;=\; q^{(1)}_{m+1,n} - q^{(2)}_{m,n}\,. \tag{3.17b}$$

2. **Level structure** $(k,\ell) = (1,0)$. Entries of matrix $Q_{m,n}$ are

$$q^{(0)}_{m,n} = \frac{1}{\Gamma}, \qquad q^{(1)}_{m,n} = \frac{u^{(1)}_{m-1,n}}{u^{(0)}_{m,n}}\, q^{(0)}_{m,n}, \qquad q^{(2)}_{m,n} = \frac{u^{(1)}_{m-1,n}u^{(2)}_{m-1,n}}{u^{(0)}_{m,n}u^{(1)}_{m,n}}\, q^{(0)}_{m,n}, \tag{3.18a}$$

where $\Gamma = u^{(1)}_{m,n}u^{(2)}_{m,n} + u^{(1)}_{m-1,n}u^{(2)}_{m-1,n} + u^{(1)}_{m-1,n}u^{(2)}_{m,n}$. The corresponding differential-difference equations are

$$\partial_t u^{(0)}_{m,n} \;=\; q^{(0)}_{m+1,n} - q^{(0)}_{m,n}\,, \quad \partial_t u^{(1)}_{m,n} \;=\; q^{(1)}_{m+1,n} - q^{(1)}_{m,n}\,, \quad \partial_t u^{(2)}_{m,n} \;=\; q^{(2)}_{m+1,n} - q^{(2)}_{m,n}\,. \tag{3.18b}$$

3. **Level structure** $(k,\ell) = (1,2)$. Entries of matrix $Q_{m,n}$ are

$$q^{(0)}_{m,n} = \frac{u^{(2)}_{m-1,n}}{u^{(1)}_{m,n}}\, q^{(1)}_{m,n}, \qquad q^{(1)}_{m,n} = \frac{1}{\Gamma}, \qquad q^{(2)}_{m,n} = \frac{u^{(1)}_{m-1,n}u^{(2)}_{m-1,n}}{u^{(0)}_{m,n}u^{(1)}_{m,n}}\, q^{(1)}_{m,n}, \tag{3.19a}$$

where $\Gamma = u^{(1)}_{m-1,n}u^{(2)}_{m-1,n} + u^{(0)}_{m,n}u^{(2)}_{m,n} + u^{(2)}_{m-1,n}u^{(2)}_{m,n}$. The corresponding differential-difference equations are

$$\partial_t u^{(0)}_{m,n} \;=\; q^{(1)}_{m+1,n} - q^{(0)}_{m,n}\,, \quad \partial_t u^{(1)}_{m,n} \;=\; q^{(2)}_{m+1,n} - q^{(1)}_{m,n}\,, \quad \partial_t u^{(2)}_{m,n} \;=\; q^{(0)}_{m+1,n} - q^{(2)}_{m,n}\,. \tag{3.19b}$$

3.2 Hierarchies of Commuting Flows

The flow (3.12) is just the first of an infinite hierarchy of (autonomous) isospectral flows. As is often the case, these can be constructed with the use of a *master symmetry*. The results of this section are summarised in Proposition 3.9.

We denote this first autonomous vector field by X^1:

$$X^1 = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{N-1} X^{1j}_{m+i,n} \partial_{u^{(j)}_{m+i,n}},$$

where this infinite sum is the *formal prolongation* of the components $X^{1j}_{m,n} = q^{(j-\ell)}_{m+1} - q^{(j)}_m$ to the shifted variables. The process is much simpler than in the case of symmetries of <u>differential</u> equations. Here, we just take the formula for the component $X^{1j}_{m,n}$ and make the <u>shift</u> $m \mapsto m+i$.

Acting on functions of a *finite* number of shifts, this sum is finite, so well defined. We denote the corresponding $t-$parameter as t^1.

Definition 3.4 (Master Symmetry) *A vector field X^M, given by*

$$X^M = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{N-1} X^{Mj}_{m+i,n} \partial_{u^{(j)}_{m+i,n}},$$

for some functions $X^{Mj}_{m,n}$ is said to be a master symmetry *of X^1 if*

$$[[X^M, X^1], X^1] = 0, \quad \textit{whilst} \;\; [X^M, X^1] \neq 0.$$

We then define X^k recursively by $X^{k+1} = [X^M, X^k]$.

Proposition 3.5 *Given the sequence of vector fields X^k, defined above, we suppose that, for some $\ell \geq 2$, $\{X^1, \dots, X^\ell\}$ pairwise commute. Then $[X^i, X^{\ell+1}] = 0$, for $1 \leq i \leq \ell - 1$.*

Remark 3.6 *This follows from an application of the Jacobi identity, but we <u>cannot</u> deduce that $[[X^M, X^\ell], X^\ell] = 0$. Since we are <u>given</u> this equality for $\ell = 2$, we <u>can</u> deduce that $[X^1, X^3] = 0$ (see the discussion around Theorem 19 of [15]). Nevertheless it <u>is</u> possible to check this by hand for low values of ℓ, for all the examples given in this paper.*

Since some of the ABS equations fall into our general class, we can generalise known results [14] on master symmetries. We thus consider two vector fields defined by matrices $S^1_{m,n}$ and $S^M_{m,n}$:

1. $S^1_{m,n}$, corresponding to the vector field X^1, defined by (3.12).

2. $S^M_{m,n}$, corresponding to the vector field X^M, defined by (3.14).

Note that $S^M_{m,n} = mS^1_{m,n}$. Then

$$\partial_\tau L_{m,n} = S^M_{m+1,n} L_{m,n} - L_{m,n} S^M_{m,n} \quad \Rightarrow \quad \partial_\tau u^{(i)}_{m,n} = (m+1) q^{(i-\ell)}_{m+1,n} - m q^{(i)}_{m,n}, \tag{3.20}$$

with $q^{(i)}_{m,n}$ defined as in the autonomous case. This defines the components of the vector field X^M. It can be checked that $[X^M, X^1] = X^2$ is nonzero and that $[X^1, X^2] = 0$, so X^M defines a master symmetry for X^1.

Consider the compatibility of the two equations

$$X^1\psi_{m,n} = S^1_{m,n}\psi_{m,n}, \quad \text{and} \quad X^M\psi_{m,n} = S^M_{m,n}\psi_{m,n}. \tag{3.21}$$

Since X^1 and X^M do not commute, we cannot consider $\psi_{m,n}$ as being *simultaneously* dependent on both t^1 and τ. The compatibility condition for these equations is

$$X^2\psi_{m,n} = [X^M, X^1]\psi_{m,n} = (X^M S^1_{m,n} - X^1 S^M_{m,n} + [S^1_{m,n}, S^M_{m,n}])\psi_{m,n} = S^2_{m,n}\psi_{m,n},$$

defining the next flow.

Definition 3.7 (The k^{th} flow) *We can recursively define the k^{th} flow by*

$$X^k\psi_{m,n} = S^k_{m,n}\psi_{m,n}, \quad \textit{where} \quad S^k_{m,n} = X^M S^{k-1}_{m,n} - X^{k-1}S^M_{m,n} + [S^{k-1}_{m,n}, S^M_{m,n}], \;\; k \geq 2. \tag{3.22}$$

Remark 3.8 (Assuming commutativity) *We know that $[X^1, X^2] = [X^1, X^3] = 0$, but for the next calculation we assume that $[X^i, X^j] = 0$, for all $i, j \geq 1$. This means that we have the* zero curvature *conditions*

$$X^j S^i_{m,n} - X^i S^j_{m,n} + [S^i_{m,n}, S^j_{m,n}] = 0. \tag{3.23}$$

Recalling that $S^M_{m,n} = mS^1_{m,n}$, the formula for $S^k_{m,n}$, using (3.23), then gives

$$S^k_{m,n} = X^M S^{k-1}_{m,n} - m(X^{k-1}S^1_{m,n} - [S^{k-1}_{m,n}, S^1_{m,n}]) = X^M S^{k-1}_{m,n} - mX^1 S^{k-1}_{m,n} = \mathcal{R}S^{k-1}_{m,n},$$

where (using (3.8b) and (3.20))

$$\mathcal{R} = \sum_{i=-\infty}^{\infty}\sum_{j=0}^{N-1}\left((i+1)q^{(j-\ell)}_{m+i+1,n} - iq^{(j)}_{m+i,n}\right)\partial_{u^{(j)}_{m+i,n}}. \tag{3.24}$$

We now define a sequence of matrices $Q^k_{m,n}$, with $S^k_{m,n} = L^{-1}_{m,n}Q^k_{m,n}$, where $Q^1_{m,n} = Q_{m,n}$ (of Proposition 3.3). We then have

$$\mathcal{R}S^k_{m,n} = L^{-1}_{m,n}\left(\mathcal{R}Q^k_{m,n} - (\mathcal{R}L_{m,n})L^{-1}_{m,n}Q^k_{m,n}\right),$$

thus giving the recursion

$$Q^{k+1}_{m,n} = \mathcal{R}Q^k_{m,n} - (\mathcal{R}L_{m,n})L^{-1}_{m,n}Q^k_{m,n} = \mathcal{R}Q^k_{m,n} - \Omega^{-\ell}Q^1_{m+1,n}\Omega^\ell L^{-1}_{m,n}Q^k_{m,n}. \tag{3.25}$$

Starting with $Q^1_{m,n}$, which is independent of λ, we find

$$Q^2_{m,n} = \mathcal{R}Q^1_{m,n} - \Omega^{-\ell}Q^1_{m+1,n}\Omega^\ell L^{-1}_{m,n}Q^1_{m,n} = \mathcal{R}Q^1_{m,n} + \lambda \text{ dependent terms.}$$

which is the sum of two parts, the first of which is independent of λ, whilst the second is rational in λ, with $\det L_{m,n}$ in the denominator. Using the recursion, we find

$$Q^k_{m,n} = \mathcal{R}^{k-1}Q^1_{m,n} + \lambda \text{ dependent terms.} \tag{3.26}$$

Again, the first part is independent of λ and the second a rational function (with $(\det L_{m,n})^{k-1}$ in the denominator).

The calculation of the evolution $\partial_{t^k}U_{m,n}$ leads to an analogous formula to (3.5):

$$\left(U_{m+1,n} + \lambda\Omega^\ell\right)\left(\partial_{t^k}U_{m,n} + Q^k_{m,n}\right) = Q^k_{m+1,n}\left(U_{m,n} + \lambda\Omega^\ell\right). \tag{3.27}$$

It follows from the structure of $Q^k_{m,n}$ that

$$\partial_{t^k}U_{m,n} = \Omega^{-\ell}\mathcal{S}_m\left(\mathcal{R}^{k-1}Q^1_{m,n}\right)\Omega^\ell - \mathcal{R}^{k-1}Q^1_{m,n}.$$

We summarise these results in:

Proposition 3.9 *The master symmetry*

$$\partial_\tau u^{(i)}_{m,n} = (m+1)q^{(i-\ell)}_{m+1,n} - mq^{(i)}_{m,n}$$

generates the hierarchy of symmetries

$$\partial_{t^k} u^{(i)}_{m,n} \;=\; \mathcal{S}_m\left(\mathcal{R}^{k-1}\left(q^{(i-\ell)}_{m,n}\right)\right) \;-\; \mathcal{R}^{k-1}\left(q^{(i)}_{m,n}\right), \tag{3.28}$$

where

$$\mathcal{R} = \sum_{i=-\infty}^{\infty}\sum_{j=0}^{N-1}\left((i+1)q^{(j-\ell)}_{m+i+1,n} - iq^{(j)}_{m+i,n}\right)\partial_{u^{(j)}_{m+i,n}},$$

and $\partial_{t^1} u^{(i)}_{m,n}$ is given by (3.12). Equation (3.28) is the compatibility condition of

$$\Psi_{m+1,n} \;=\; L_{m,n}\Psi_{m,n}\,, \quad \partial_{t^k}\Psi_{m,n} \;=\; S^k_{m,n}\,\Psi_{m,n},$$

where $S^k_{m,n}$ is defined recursively by $S^k_{m,n} = \mathcal{R}S^{k-1}_{m,n}$.

4 Symmetries of the Difference Equations

In Section 3 we considered the compatibility of a discrete shift in the $m-$direction and a continuous evolution in t, given by (3.1). The simplest case is described in Proposition 3.3. The master symmetry of Section 3.2 generates an infinite family of commuting symmetries.

The compatibility of discrete shifts in both $m-$ and $n-$directions (equations (2.1)) leads to our fully discrete system (2.4) (written explicitly as (2.5)). We now consider the flows (3.28) as symmetries of this discrete system.

4.1 The Evolution $\partial_t\, v^{(i)}_{m,n}$

For any time evolution (3.1), the compatibility of

$$\Psi_{m,n+1} = M_{m,n}\Psi_{m,n}, \quad \partial_t\Psi_{m,n} = S_{m,n}\,\Psi_{m,n}, \tag{4.1a}$$

with $M_{m,n}$ defined by (2.1b), gives the equation

$$\partial_t\, M_{m,n} = S_{m,n+1}\, M_{m,n} - M_{m,n}\, S_{m,n}. \tag{4.1b}$$

Here $S_{m,n} = L^{-1}_{m,n}Q_{m,n}$, where $Q_{m,n}$ and $U_{m,n}$ must satisfy (3.5). Using (3.2) and (4.1b), we then have

$$\begin{aligned}\partial_t(L_{m,n+1}M_{m,n} - M_{m+1,n}L_{m,n}) \;&=\; S_{m+1,n+1}(L_{m,n+1}M_{m,n} - M_{m+1,n}L_{m,n})\\ &\qquad -(L_{m,n+1}M_{m,n} - M_{m+1,n}L_{m,n})S_{m,n},\end{aligned}$$

showing compatibility on solutions of the fully discrete system (2.4).

For the flow given in Proposition 3.3, we find

$$L_{m,n+1}\partial_t M_{m,n} = Q_{m,n+1}\, M_{m,n} - L_{m,n+1}\, M_{m,n}\, L^{-1}_{m,n}\, Q_{m,n} = Q_{m,n+1}\, M_{m,n} - M_{m+1,n}\, Q_{m,n},$$

where we used the difference equation (2.3).

The explicit forms of L and M then lead to

$$\begin{aligned} \partial_t V_{m,n} &= \Omega^{-\ell_1} Q_{m,n+1} \Omega^{\ell_2} - \Omega^{\ell_2-\ell_1} Q_{m,n}, \\ U_{m,n+1}\, \partial_t V_{m,n} &= Q_{m,n+1}\, V_{m,n} - V_{m+1,n}\, Q_{m,n}. \end{aligned}$$

The first of these is just the $t-$evolution of $V_{m,n}$, which, in components, is just

$$\partial_t v^{(i)}_{m,n} = q^{(i-\ell_1)}_{m,n+1} - q^{(i+\ell_2-\ell_1)}_{m,n}, \tag{4.2a}$$

whilst the second, using (2.4b) to eliminate $V_{m+1,n}$, leads to

$$\Omega^{\ell_1} V_{m,n} \Omega^{-\ell_1} Q_{m,n} - Q_{m,n+1} V_{m,n} = \Omega^{\ell_2} U_{m,n} \Omega^{-\ell_1} Q_{m,n} - U_{m,n+1} \Omega^{-\ell_1} Q_{m,n+1} \Omega^{\ell_2}. \tag{4.2b}$$

Remark 4.1 *Each expression in this equation has level $k_1 + k_2$ (requiring the condition (2.2)). Since U, V and Q are* known quantities*, this looks like an additional constraint, but it can be shown that this holds identically as a consequence of previous equations.*

Similar results can be calculated for the $\tau-$flow (3.14) (see Proposition 4.2). We can extend this discussion to the flows introduced in Section 3.2. The analogous formula to (3.28) is

$$\partial_{t^k} v^{(i)}_{m,n} = \mathcal{S}_n\left(\mathcal{R}^{k-1}\left(q^{(i-\ell_1)}_{m,n}\right)\right) - \mathcal{R}^{k-1}\left(q^{(i+\ell_2-\ell_1)}_{m,n}\right). \tag{4.3}$$

The remaining parts of (4.1b) would give several conditions analogous to (4.2b). We conjecture that these hold identically, but have no general proof. For all specific examples calculated, this is the case.

4.2 Symmetries in the $n-$Direction

This whole structure can be repeated for continuous flows in the $n-$direction:

$$\partial_s \Psi_{m,n} = (V_{m,n} + \lambda \Omega^{\ell_2})^{-1} R_{m,n} \Psi_{m,n}, \tag{4.4}$$

with the simplest choice being that $R_{m,n}$ is $\lambda-$independent. The analogous formulae to (3.6) are

$$R_{m,n} V_{m,n-1} - V_{m,n} \Omega^{-\ell_2} R_{m,n} \Omega^{\ell_2} = 0 \quad \text{and} \quad \partial_s V_{m,n} = \Omega^{-\ell_2} R_{m,n+1} \Omega^{\ell_2} - R_{m,n+1}.$$

The results following from the mutual compatibility of the four linear equations are summarised in the proposition below.

Proposition 4.2 *Let $(k_1, \ell_1\,;\, k_2, \ell_2)$ satisfy (2.2), with $(N, k_1 - \ell_1) = (N, k_2 - \ell_2) = 1$, and consider the system of equations*

$$\Psi_{m+1,n} = \left(U_{m,n} + \lambda \Omega^{\ell_1}\right) \Psi_{m,n}\,, \tag{4.5a}$$

$$\Psi_{m,n+1} = \left(V_{m,n} + \lambda \Omega^{\ell_2}\right) \Psi_{m,n}\,, \tag{4.5b}$$

$$\partial_t \Psi_{m,n} = \left(U_{m,n} + \lambda \Omega^{\ell_1}\right)^{-1} Q_{m,n} \Psi_{m,n}\,, \tag{4.5c}$$

$$\partial_s \Psi_{m,n} = \left(V_{m,n} + \lambda \Omega^{\ell_2}\right)^{-1} R_{m,n} \Psi_{m,n}\,, \tag{4.5d}$$

where

$$U_{m,n} = \operatorname{diag}\left(u^{(0)}_{m,n}, \cdots, u^{(N-1)}_{m,n}\right)\Omega^{k_1}, \quad V_{m,n} = \operatorname{diag}\left(v^{(0)}_{m,n}, \cdots, v^{(N-1)}_{m,n}\right)\Omega^{k_2}, \tag{4.6}$$

$$Q_{m,n} = \operatorname{diag}\left(q^{(0)}_{m,n}, \cdots, q^{(N-1)}_{m,n}\right)\Omega^{k_1}, \quad R_{m,n} = \operatorname{diag}\left(r^{(0)}_{m,n}, \cdots, r^{(N-1)}_{m,n}\right)\Omega^{k_2}. \tag{4.7}$$

Then, the system of difference equations (2.5) follows from the compatibility condition of equations (4.5a) and (4.5b).
The differential-difference equations

$$\partial_t u^{(i)}_{m,n} = q^{(i-\ell_1)}_{m+1,n} - q^{(i)}_{m,n}\,, \qquad \partial_t v^{(i)}_{m,n} = q^{(i-\ell_1)}_{m,n+1} - q^{(i+\ell_2-\ell_1)}_{m,n}\,, \tag{4.8a}$$

$$\partial_\tau u^{(i)}_{m,n} = (m+1)q^{(i-\ell_1)}_{m+1,n} - mq^{(i)}_{m,n}\,, \qquad \partial_\tau v^{(i)}_{m,n} = m(q^{(i-\ell_1)}_{m,n+1} - q^{(i+\ell_2-\ell_1)}_{m,n})\,, \tag{4.8b}$$

where $i \in \mathbb{Z}_N$ and the functions $q^{(i)}_{m,n}$ are solutions of

$$q^{(i)}_{m,n}\, u^{(i+k_1)}_{m-1,n} = u^{(i)}_{m,n}\, q^{(i+k_1-\ell_1)}_{m,n} \quad \text{and} \quad \sum_{i=0}^{N-1} \frac{q^{(i)}_{m,n}}{u^{(i)}_{m,n}} = \frac{1}{a}, \tag{4.9}$$

follow from the compatibility condition of equation (4.5c) with (4.5a) and (4.5b), respectively, and define symmetries of system (2.5) in the m direction. The $\tau-$flow satisfies $\partial_\tau a = 1$, $\partial_\tau b = 0$.
The differential-difference equations

$$\partial_s u^{(i)}_{m,n} = r^{(i-\ell_2)}_{m+1,n} - r^{(i+\ell_1-\ell_2)}_{m,n}\,, \qquad \partial_s v^{(i)}_{m,n} = r^{(i-\ell_2)}_{m,n+1} - r^{(i)}_{m,n}\,, \tag{4.10a}$$

$$\partial_\sigma u^{(i)}_{m,n} = n(r^{(i-\ell_2)}_{m+1,n} - r^{(i+\ell_1-\ell_2)}_{m,n})\,, \qquad \partial_\sigma v^{(i)}_{m,n} = (n+1)r^{(i-\ell_2)}_{m,n+1} - nr^{(i)}_{m,n}\,, \tag{4.10b}$$

where $i \in \mathbb{Z}_N$ and the functions $r^{(i)}_{m,n}$ are solutions of

$$r^{(i)}_{m,n}\, v^{(i+k_2)}_{m,n-1} = v^{(i)}_{m,n}\, r^{(i+k_2-\ell_2)}_{m,n} \quad \text{and} \quad \sum_{i=0}^{N-1} \frac{r^{(i)}_{m,n}}{v^{(i)}_{m,n}} = \frac{1}{b}, \tag{4.11}$$

follow from the compatibility condition of equation (4.5d) with (4.5a) and (4.5b), respectively, and define symmetries of system (2.5) in the n direction. The $\sigma-$flow satisfies $\partial_\sigma a = 0$, $\partial_\sigma b = 1$.

Remark 4.3 (Master Symmetries) *The non-autonomous flows (4.8b) and (4.10b) are master symmetries for their respective hierarchies.*

4.3 Symmetries in Potentials Variables

We can explicitly write these symmetries in terms of the potentials introduced in Section 2.2.

4.3.1 Quotient Potentials

Substituting $u^{(i)}_{m,n} = \alpha \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}}$ into the first of (4.8a), we obtain

$$\alpha\, \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}} \left(\frac{\partial_t \phi^{(i)}_{m+1,n}}{\phi^{(i)}_{m+1,n}} - \frac{\partial_t \phi^{(i+k_1)}_{m,n}}{\phi^{(i+k_1)}_{m,n}} \right) = q^{(i-\ell_1)}_{m+1,n} - q^{(i)}_{m,n}.$$

Using the potential form of (3.8a),

$$\frac{\phi^{(i+k_1)}_{m,n}}{\phi^{(i)}_{m+1,n}}\, q^{(i)}_{m,n} = \frac{\phi^{(i+2k_1)}_{m-1,n}}{\phi^{(i+k_1)}_{m,n}}\, q^{(i+k_1-\ell_1)}_{m,n},$$

we can write this as

$$\alpha\,\frac{\partial_t\phi^{(i+k_1)}_{m,n}}{\phi^{(i+k_1)}_{m,n}} - \frac{\phi^{(i+2k_1)}_{m-1,n}}{\phi^{(i+k_1)}_{m,n}}\, q^{(i+k_1-\ell_1)}_{m,n} = \alpha\,\frac{\partial_t\phi^{(i)}_{m+1,n}}{\phi^{(i)}_{m+1,n}} - \frac{\phi^{(i+k_1)}_{m,n}}{\phi^{(i)}_{m+1,n}}\, q^{(i-\ell_1)}_{m+1,n}. \tag{4.12}$$

Since the right side of (4.12) is just the left side with $(m,i)\mapsto(m+1,i-k_1)$, we have

$$\frac{\partial_t\phi^{(i)}_{m,n}}{\phi^{(i)}_{m,n}} = \frac{\phi^{(i+k_1)}_{m-1,n}}{\alpha\phi^{(i)}_{m,n}}\, q^{(i-\ell_1)}_{m,n} + c, \tag{4.13}$$

where c is a constant, to be determined. Since

$$\partial_t\left(\log\prod_{i=0}^{N-1}\phi^{(i)}_{m,n}\right) = \sum_{i=0}^{N-1}\left(\frac{\phi^{(i+k_1)}_{m-1,n}}{\alpha\phi^{(i)}_{m,n}}\, q^{(i-\ell_1)}_{m,n}\right) + cN = 0,$$

then

$$cN = -\sum_{i=0}^{N-1}\frac{q^{(i-\ell_1)}_{m,n}}{u^{(i)}_{m-1,n}} = -\sum_{i=0}^{N-1}\frac{q^{(i-k_1)}_{m,n}}{u^{(i-k_1)}_{m,n}} = -\frac{1}{\alpha^N}.$$

Carrying out a similar calculation of the $s-$symmetry (4.10a), we obtain the formulae

$$\partial_t\,\phi^{(i)}_{m,n} = \alpha^{-1}\, q^{(i-\ell_1)}_{m,n}\,\phi^{(i+k_1)}_{m-1,n} - \frac{\phi^{(i)}_{m,n}}{N\alpha^N}\,, \tag{4.14a}$$

$$\partial_s\,\phi^{(i)}_{m,n} = \beta^{-1}\, q^{(i-\ell_2)}_{m,n}\,\phi^{(i+k_2)}_{m,n-1} - \frac{\phi^{(i)}_{m,n}}{N\beta^N}. \tag{4.14b}$$

Remark 4.4 *These choices of constants have made the vector fields* tangent *to the level surfaces* $\prod_{i=0}^{N-1}\phi^{(i)}_{m,n} =$ *constant, so these symmetries survive the reduction to* $N-1$ *components, which we always make in our examples.*

The *master symmetries* are calculated in the same way, but we must take into account that $\partial_\tau\alpha = 1/(N\alpha^{N-1})$ and $\partial_\sigma\beta = 1/(N\beta^{N-1})$, which implies that the additional constants must depend upon m and n respectively, giving

$$\partial_\tau\,\phi^{(i)}_{m,n} = m\,\alpha^{-1}\, q^{(i-\ell_1)}_{m,n}\,\phi^{(i+k_1)}_{m-1,n} - \frac{m\,\phi^{(i)}_{m,n}}{N\alpha^N}\,, \tag{4.15a}$$

$$\partial_\sigma\,\phi^{(i)}_{m,n} = n\,\beta^{-1}\, q^{(i-\ell_2)}_{m,n}\,\phi^{(i+k_2)}_{m,n-1} - \frac{n\,\phi^{(i)}_{m,n}}{N\beta^N}. \tag{4.15b}$$

Example 4.5 ($(k_1,\ell_1;k_2,\ell_2)=(1,2;1,2)$) Using (2.9b), we can make the replacement

$$\left(\phi^{(0)}_{m,n}\,,\,\phi^{(1)}_{m,n}\,,\,\phi^{(2)}_{m,n}\right) \mapsto \left(\frac{1}{\phi^{(0)}_{m,n}}\,,\,\phi^{(1)}_{m,n}\,,\,\frac{\phi^{(0)}_{m,n}}{\phi^{(1)}_{m,n}}\right),$$

to get the 2−component system

$$
\begin{aligned}
\phi^{(0)}_{m+1,n+1} &= \left(\frac{\alpha\,\phi^{(1)}_{m+1,n}-\beta\phi^{(1)}_{m,n+1}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}-\beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(0)}_{m,n}}\,,\\
\phi^{(1)}_{m+1,n+1} &= \left(\frac{\alpha\,\phi^{(0)}_{m,n+1}-\beta\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}-\beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(1)}_{m,n}}\,,
\end{aligned}
\tag{4.16}
$$

which admits two point symmetries generated by

$$\partial_\epsilon\phi^{(0)}_{m,n}=\omega^{n+m}\phi^{(0)}_{m,n},\ \ \partial_\epsilon\phi^{(1)}_{m,n}=0\quad\text{and}\quad\partial_\eta\phi^{(0)}_{m,n}=0,\ \ \partial_\eta\phi^{(1)}_{m,n}=\omega^{n+m}\phi^{(1)}_{m,n},$$

where $\omega^2+\omega+1=0$. Written in terms of these potentials, the symmetries (3.19) take the form

$$\partial_{t^1}\phi^{(0)}_{m,n}=-\frac{\phi^{(0)}_{m,n}}{\alpha^3}\left(\frac{\gamma^{(1)}_{m,n}}{\mathcal{F}_{m,n}}-\frac{1}{3}\right),\quad \partial_{t^1}\phi^{(1)}_{m,n}=\frac{\phi^{(1)}_{m,n}}{\alpha^3}\left(\frac{\gamma^{(0)}_{m,n}}{\mathcal{F}_{m,n}}-\frac{1}{3}\right),\tag{4.17}$$

where $\gamma^{(i)}_{m,n}=\phi^{(i)}_{m+1,n}\phi^{(i)}_{m,n}\phi^{(i)}_{m-1,n}$ and $\mathcal{F}_{m,n}=1+\gamma^{(0)}_{m,n}+\gamma^{(1)}_{m,n}$.

The local master symmetry

$$\partial_\tau\phi^{(0)}_{m,n}\;=\;m\,\partial_{t^1}\phi^{(0)}_{m,n}\,,\quad \partial_\tau\phi^{(1)}_{m,n}\;=\;m\,\partial_{t^1}\phi^{(1)}_{m,n}\,,\quad \partial_\tau\alpha\;=\;\frac{1}{3\alpha^2},$$

allows us to construct a hierarchy of symmetries of system (4.16) in the m-direction. The formula $[X^M,X^1]=X^2$ gives a linear combination of a genuinely new symmetry and X^1. The "new" part of X^2 is given by

$$
\begin{aligned}
\partial_{t^2}\phi^{(0)}_{m,n}&=\frac{\phi^{(0)}_{m,n}}{\alpha^6}\,\frac{\gamma^{(1)}_{m,n}}{\mathcal{F}_{m,n}}\,(\mathcal{S}_m+1)\left(\frac{\Delta_m\left(\gamma^{(0)}_{m-1,n}\right)}{\mathcal{F}_{m,n}\mathcal{F}_{m-1,n}}\right),\\
\partial_{t^2}\phi^{(1)}_{m,n}&=\frac{\phi^{(1)}_{m,n}}{\alpha^6}\,\frac{\gamma^{(0)}_{m,n}}{\mathcal{F}_{m,n}}\,(\mathcal{S}_m+1)\left(\frac{\Delta_m\left(\gamma^{(1)}_{m-1,n}\right)}{\mathcal{F}_{m,n}\mathcal{F}_{m-1,n}}\right).
\end{aligned}
$$

Similar considerations hold for symmetries in the n-direction. They actually follow from the above ones by employing the invariance of system (4.16) under the interchange of lattice variables and parameters.

4.3.2 Additive Potentials

We consider symmetries of the discrete system (2.10b). An obvious Lie point symmetry is $\partial_\varepsilon\chi^{(i)}_{m,n}=1$. In terms of the potentials $\chi^{(i)}$, defined by (2.10a), the differential-difference systems (4.8a) and (4.10a) take the form

$$\partial_t\,\chi^{(i)}_{m,n}\;=\;q^{(i-\ell_1)}_{m,n}\,,\quad \partial_s\,\chi^{(i)}_{m,n}\;=\;r^{(i-\ell_2)}_{m,n}\,,\quad i\in\mathbb{Z}_N\,.$$

Example 4.6 (Equivalence class $(k_1,\ell_2;k_2,\ell_2)=(0,1;0,1)$ **)** A Discrete Boussinesq System, introduced in [7], is

$$
\begin{aligned}
\chi^{(0)}_{m+1,n+1} &= \frac{(\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n})\chi^{(1)}_{m+1,n}-(\chi^{(0)}_{m,n+1}-\chi^{(1)}_{m,n})\chi^{(1)}_{m,n+1}}{\chi^{(0)}_{m+1,n}-\chi^{(0)}_{m,n+1}},\\
\chi^{(1)}_{m+1,n+1} &= \chi^{(0)}_{m,n}+\frac{1}{\chi^{(1)}_{m+1,n}-\chi^{(1)}_{m,n+1}}\left(\frac{\alpha^3}{\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n}}-\frac{\beta^3}{\chi^{(0)}_{m,n+1}-\chi^{(1)}_{m,n}}\right).
\end{aligned}
\tag{4.18}
$$

The lowest order symmetries for this system are generated by

$$\partial_{t^1}\chi^{(0)}_{m,n} = \frac{\chi^{(0)}_{m,n}-\chi^{(1)}_{m-1,n}}{\alpha^3+(\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n})(\chi^{(0)}_{m,n}-\chi^{(1)}_{m-1,n})(\chi^{(1)}_{m+1,n}-\chi^{(0)}_{m-1,n})},$$

$$\partial_{t^1}\chi^{(1)}_{m,n} = \frac{\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n}}{\alpha^3+(\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n})(\chi^{(0)}_{m,n}-\chi^{(1)}_{m-1,n})(\chi^{(1)}_{m+1,n}-\chi^{(0)}_{m-1,n})},$$

and a master symmetry is

$$\partial_\tau\chi^{(i)}_{m,n} = m\,\partial_{t^1}\chi^{(i)}_{m,n}\,,\quad \partial_\tau\alpha = \frac{1}{3\alpha^2}\,,\quad i=0,1.$$

5 Symmetries in the Degenerate Case

In this section we consider the reduction of the symmetries of our fully discrete system to the degenerate case (see Section 2.1). Whereas the $t-$flow (4.8a) can always be reduced to this subcase, the $s-$flow (4.10a) presents difficulties.

Analogous to the formula (3.3), we have

$$\partial_s\log(\det M_{m,n}) = \Delta_n\left(\mathrm{Tr}\left(M^{-1}_{m,n}R_{m,n}\right)\right). \tag{5.1}$$

For the generic case ($b\neq 0$) we can calculate the formula which is analogous to (3.9)

$$\partial_s(b-(-\lambda)^N) = b\,\Delta_n\left(\sum_{i=0}^{N-1}\frac{r^{(i)}_{m,n}}{v^{(i)}_{m,n}}\right) = \Delta_n\left(\sum_{i=0}^{N-1} r^{(i)}_{m,n}\prod_{j\neq i} v^{(j)}_{m,n}\right) \tag{5.2}$$

If we now let $v^{(N-1)}_{m,n}\to 0$ (so $b\to 0$), then the left hand side vanishes, leading to

$$r^{(N-1)}_{m,n}\prod_{i=0}^{N-2} v^{(i)}_{m,n} = \text{constant}. \tag{5.3a}$$

If the above product is nonzero, then we can set this constant to be 1. Otherwise, this equation trivialises.

In order to define the symmetry (4.10a), we need to solve the equations

$$r^{(i)}_{m,n}\,v^{(i+k_2)}_{m,n-1} = v^{(i)}_{m,n}\,r^{(i+k_2-\ell_2)}_{m,n},\quad i\in\mathbb{Z}_N, \tag{5.3b}$$

with $v^{(N-1)}_{m,n}=0$, in conjunction with (5.3a).

This symmetry can only be constructed in the case $v^{(N-1)}=0$, $v^{(i)}\neq 0, i\neq N-1$. In all other cases there are not enough equations to determine $r^{(i)}$. Furthermore, if $k_2\neq 0$, then successive equations imply that successive components $r^{(i)}_{m,n}$ must vanish (or that further $v^{(i)}$ components must vanish). This can lead to either a trivial symmetry (all components are zero) or to arbitrary functions. This phenomenon occurs in Example 5.3 below.

At the other extreme, $v^{(0)}\neq 0$, $v^{(i)}=0, i\neq 0$, we can also calculate a symmetry, using another approach, which is described in Section 5.2.

5.1 The Case $v^{(N-1)} = 0$, $v^{(i)} \neq 0$, $i \neq N-1$

We present some examples for $N = 2$ and $N = 3$, to illustrate the way of solving Equations (5.3).

Example 5.1 (Hirota's KdV Equation: $N = 2$, equivalence class $(0,1;0,1)$) Setting $v^{(1)}_{m,n} = 0$ in equations (2.5) gives us

$$u^{(0)}_{m,n} = u_{m,n}, \quad u^{(1)}_{m,n} = \frac{a}{u_{m,n}}, \quad v^{(0)}_{m,n} = u_{m,n} - \frac{a}{u_{m,n+1}},$$

and Hirota's KdV equation

$$\frac{a}{u_{m+1,n+1}} + u_{m,n+1} = u_{m+1,n} + \frac{a}{u_{m,n}}. \tag{5.4}$$

Symmetries for this equation have been discussed by Yamilov [15]. In our framework, the $t-$flows for this are direct reductions of those for the 2−component system. The first symmetry (3.15) reduces to the single component

$$\partial_{t^1} u_{m,n} = u_{m,n}\Delta_m\left(\frac{1}{u_{m,n}u_{m-1,n}+a}\right).$$

Since the $t-$hierarchy is impervious to the degeneration of V, the corresponding master symmetry survives:

$$\partial_\tau u_{m,n} = u_{m,n}\Delta_m\left(\frac{m}{u_{m,n}u_{m-1,n}+a}\right), \quad \partial_\tau a = 1.$$

For the s^1-flow, we note that

$$r^{(0)}_{m,n} = \frac{1}{v^{(0)}_{m,n-1}}, \; r^{(1)}_{m,n} = \frac{1}{v^{(0)}_{m,n}} \quad \Rightarrow \quad \partial_{s^1} v^{(0)}_{m,n} = \frac{1}{v^{(0)}_{m,n+1}} - \frac{1}{v^{(0)}_{m,n-1}}, \tag{5.5}$$

using (4.10a). For the variable $u_{m,n}$, defined above, we have

$$\partial_{s^1} u_{m,n} = \frac{1}{v^{(0)}_{m+1,n}} - \frac{1}{v^{(0)}_{m,n-1}} = u_{m,n}\Delta_n\left(\frac{1}{u_{m,n}u_{m,n-1}-a}\right), \tag{5.6}$$

using (4.10a), the above formula for $v^{(0)}_{m,n}$ and Hirota's equation (5.4) (to replace $u_{m+1,n+1}$). Not all s^i symmetries survive this reduction, indicating that the 2−component master symmetry does not reduce. However, since equation (5.4) is invariant under a simple discrete symmetry $(m,n,a) \mapsto (n,m,-a)$ (under which the t^1 and s^1 flows interchange), we can write the master symmetry for the $s-$hierarchy:

$$\partial_\sigma u_{m,n} = u_{m,n}\Delta_n\left(\frac{n}{u_{m,n}u_{m,n-1}-a}\right), \quad \partial_\sigma a = -1.$$

To derive this from the reduced spectral problem requires the consideration of non-isospectral flows.

Example 5.2 ($N = 3$, equivalence class $(0,1;0,1)$) In [7] we give a 2−component generalisation of Hirota's KdV equation. Now we set $v^{(2)}_{m,n} = 0$ in equations (2.5) to obtain

$$u^{(2)}_{m,n} = \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n}}, \quad v^{(0)}_{m,n} = u^{(0)}_{m,n} - \frac{a}{u^{(0)}_{m,n+1}u^{(1)}_{m,n+1}}, \quad v^{(1)}_{m,n} = u^{(1)}_{m,n} - \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n+1}},$$

together with the system

$$\frac{a}{u^{(0)}_{m+1,n+1}u^{(1)}_{m+1,n+1}} + u^{(0)}_{m,n+1} = u^{(0)}_{m+1,n} + \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n+1}}\,, \tag{5.7a}$$

$$\frac{a}{u^{(0)}_{m+1,n}u^{(1)}_{m+1,n+1}} + u^{(1)}_{m,n+1} = u^{(1)}_{m+1,n} + \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n}}\,. \tag{5.7b}$$

The t^1 flow is exactly (3.17), whilst the s^1 flow is given by $\partial_{s^1}u^{(i)}_{m,n} = r^{(i-1)}_{m+1,n} - r^{(i)}_{m,n}$, where

$$r^{(0)}_{m,n} = \frac{1}{v^{(0)}_{m,n-1}v^{(1)}_{m,n}},\quad r^{(1)}_{m,n} = \frac{1}{v^{(0)}_{m,n-1}v^{(1)}_{m,n-1}},\quad r^{(2)}_{m,n} = \frac{1}{v^{(0)}_{m,n}v^{(1)}_{m,n}},$$

which, after replacing $v^{(i)}_{m,n}$ with the above formulae and using the difference equation (5.7), can be written

$$\begin{aligned}
\partial_{s^1}u^{(0)}_{m,n} &= \frac{\left(u^{(0)}_{m,n}\right)^2 u^{(1)}_{m,n}u^{(1)}_{m,n+1}}{u^{(0)}_{m,n}u^{(1)}_{m,n}u^{(1)}_{m,n+1}-a}\Delta_n\left(\frac{1}{u^{(0)}_{m,n-1}u^{(0)}_{m,n}u^{(1)}_{m,n}-a}\right),\\
\partial_{s^1}u^{(1)}_{m,n} &= \frac{u^{(0)}_{m,n-1}u^{(0)}_{m,n}\left(u^{(1)}_{m,n}\right)^2}{u^{(0)}_{m,n-1}u^{(0)}_{m,n}u^{(1)}_{m,n}-a}\Delta_n\left(\frac{1}{u^{(0)}_{m,n-1}u^{(1)}_{m,n-1}u^{(1)}_{m,n}-a}\right).
\end{aligned}$$

The master symmetry for the $t-$hierarchy is given by

$$\partial_\tau u^{(0)}_{m,n} = u^{(0)}_{m,n}\Delta_m\left(\frac{mu^{(1)}_{m-1,n}}{\Gamma}\right),\quad \partial_\tau u^{(1)}_{m,n} = u^{(1)}_{m,n}\Delta_m\left(\frac{mu^{(0)}_{m,n}}{\Gamma}\right),$$

where $\Gamma = a(u^{(0)}_{m,n}+u^{(1)}_{m-1,n}) + u^{(0)}_{m,n}u^{(0)}_{m-1,n}u^{(1)}_{m,n}u^{(1)}_{m-1,n}$, and we have $\partial_\tau a = 1$. This time we have no simple symmetry and no master symmetry to generate a hierarchy of $s-$flows.

Equation (5.7) admits a further reduction, with $v^{(1)}_{m,n} = 0$. Whilst the t^i hierarchy survives reduction, the $s-$flows do not (the above formulae for $r^{(i)}_{m,n}$ do not allow $v^{(1)}_{m,n}\to 0$), so we leave the discussion until Section 5.2.

Example 5.3 (The equivalence class $(0,1;1,2)$, for $N=3$ and $v^{(2)}_{m,n}=0$) In this case we introduce variables $u_{m,n}$ and $v_{m,n}$ by

$$u^{(0)}_{m,n} = u_{m,n}v_{m,n},\quad u^{(1)}_{m,n} = \frac{1}{u_{m,n}},\quad u^{(2)}_{m,n} = \frac{a}{v_{m,n}}\,,$$

$$v^{(0)}_{m,n} = \frac{1}{u_{m,n}} - \frac{a}{v_{m,n+1}},\quad v^{(1)}_{m,n} = a\left(\frac{1}{v_{m,n}} - u_{m,n}u_{m,n+1}\right),$$

to derive the system

$$u_{m,n}v_{m,n} + \frac{a}{v_{m+1,n}} = \frac{1}{u_{m,n+1}} + au_{m+1,n}\,u_{m+1,n+1}\,, \tag{5.8a}$$

$$u_{m,n+1}v_{m,n+1} + \frac{a}{v_{m+1,n+1}} = \frac{1}{u_{m+1,n}} + au_{m,n}\,u_{m,n+1}\,. \tag{5.8b}$$

The symmetry in the $m-$direction is given by (3.17), but under this reduction:

$$\partial_t u = -u_{m,n}\Delta_m\left(\frac{u_{m-1,n}u_{m,n}v_{m,n}}{\Gamma}\right),\quad \partial_t v = -v_{m,n}\Delta_m\left(\frac{u_{m-1,n}v_{m-1,n}v_{m,n}}{\Gamma}\right),$$

where $\Gamma = u_{m-1,n}v_{m,n}(au_{m,n}+v_{m-1,n})+a$. On the other hand, since $k_2=1$, equations (5.3b) imply that $r^{(0)}_{m,n}=r^{(2)}_{m,n}=0$, with no constraint on $r^{(1)}_{m,n}$, leading to

$$\partial_s u_{m,n}=0\quad\text{and}\quad \partial_s v_{m,n}=v^2_{m,n}r^{(1)}_{m,n}.$$

Consistency with equation (5.8) implies that $r^{(1)}_{m,n}$ must satisfy the *difference equation*

$$r^{(1)}_{m+1,n}=u_{m,n}v^2_{m,n}r^{(1)}_{m,n},$$

so the symmetry is *nonlocal*.

Noting that the left hand sides of equations (5.8) are related by a shift in the n direction, we can derive an equation for the single component u:

$$au_{m+1,n+1}u_{m+1,n+2}+\frac{1}{u_{m,n+2}}=\frac{1}{u_{m+1,n}}+au_{m,n}u_{m,n+1}\,. \tag{5.9}$$

Equation (5.8a) is then a first order, "driven" difference equation for $v_{m,n}$. In this case the $t-$symmetry reduces to

$$\partial_t u_{m,n}=-u_{m,n}\Delta_m\left(\frac{u_{m-1,n}u_{m,n}u_{m-1,n+1}}{1+u_{m,n}u_{m-1,n+1}(u_{m-1,n}+u_{m,n+1})}\right). \tag{5.10}$$

On the other hand, the $s-$symmetry trivialises, with $r^{(1)}_{m,n}=0$.

5.2 The Case $v^{(0)}\neq 0$, $v^{(i)}=0$, $i\neq 0$

Proposition 4.2 states that for the non-degenerate case ($b\neq 0$), the system of difference equations (2.5) has the symmetry (4.10a), where $r^{(i)}_{m,n}$ are given by the solution of (4.11). In the specific *degenerate case* for which $v^{(N-1)}=0$, $v^{(i)}\neq 0$, $i\neq N-1$ the last equation of (4.11) is replaced by (5.3a). In all other cases, equations (4.11) do not determine the symmetry, so, in the present case, we replace the evolution (4.5d) by

$$\partial_{s^1}\Psi_{m,n}=W_{m,n}\Psi_{m,n},\quad\text{where}\quad W_{m,n}=-\sum_{i=1}^{N}\frac{1}{\lambda^i}W^{(i)}_{m,n}, \tag{5.11a}$$

where matrices $W^{(i)}$ are λ-independent. The compatibility of (5.11a) with the difference equation (4.5b) leads to

$$\partial_{s^1}V_{m,n}+\left(V_{m,n}+\lambda^{\ell_2}\Omega\right)W_{m,n}=W_{m,n+1}\left(V_{m,n}+\lambda^{\ell_2}\Omega\right). \tag{5.11b}$$

Collecting different powers of λ in the compatibility condition we find the relations

$$\partial_{s^1}V_{m,n}+W^{(1)}_{m,n+1}\Omega^{\ell_2}=\Omega^{\ell_2}W^{(1)}_{m,n}, \tag{5.11c}$$

$$V_{m,n}W^{(i)}_{m,n}+\Omega^{\ell_2}W^{(i+1)}_{m,n}=W^{(i)}_{m,n+1}V_{m,n}+W^{(i+1)}_{m,n+1}\Omega^{\ell_2},\quad i=1,\dots,N-1, \tag{5.11d}$$

$$V_{m,n}W^{(N)}_{m,n}=W^{(N)}_{m,n+1}V_{m,n}, \tag{5.11e}$$

Since $lev(V_{m,n}) = k_2$, we see that $lev\left(W^{(i)}_{m,n}\right) = i\,(k_2 - \ell_2)$.

We may start with a solution of equation (5.11e), noting that $lev(W^{(N)}) = 0$, and then proceed to the next equation which involves $W^{(N)}$ and $W^{(N-1)}$, and so on so forth. For generic k_2 and ℓ_2 this is quite involved but for the case $(k_2, \ell_2) = (0,1)$ where $V_{m,n}$ has only one non-zero entry, this can be done systematically.

We can now take our solution for W and consider the compatibility of (5.11a) with the difference equation (4.5a), giving

$$\partial_{s^1} U_{m,n} + \left(U_{m,n} + \lambda^{\ell_1}\Omega\right) W_{m,n} = W_{m+1,n}\left(U_{m,n} + \lambda^{\ell_1}\Omega\right). \tag{5.12}$$

5.2.1 With Level Structure $(0,1;0,1)$

To solve Equations (5.11c) - (5.11e), we choose $W^{(N)}_{m,n} = \mathrm{diag}\,(1,0,\ldots,0)$ and then recursively solve for the remaining matrices. Eventually, we find, from (5.11c), that

$$\partial_s v^{(0)}_{m,n} = w^{11}_{m,n} - w^{10}_{m,n+1}, \quad \text{where} \quad W^{(1)}_{m,n} = \mathrm{diag}\left(w^{10}_{m,n}, w^{11}_{m,n}, \ldots, w^{1N-1}_{m,n}\right)\,\Omega^{N-1}. \tag{5.13a}$$

Then (5.12) gives

$$\partial_s u^{(i)}_{m,n} = w^{1i+1}_{m,n} - w^{1i}_{m+1,n}. \tag{5.13b}$$

When $N = 2$, these formulae just give the same result as (5.5) for Hirota's KdV equation.

Example 5.4 (The Case $N = 3$) We now consider the further reduction of Equation (5.7), with $v^{(1)}_{m,n} = 0$, corresponding to $u^{(0)}_{m,n} = \frac{a}{u^{(1)}_{m,n}u^{(1)}_{m,n+1}}$. In this case, (5.7b) holds identically, whilst (5.7a) takes the form of a six point equation

$$u^{(1)}_{m,n} + \frac{a}{u^{(1)}_{m+1,n}u^{(1)}_{m+1,n+1}} = u^{(1)}_{m+1,n+2} + \frac{a}{u^{(1)}_{m,n+1}u^{(1)}_{m,n+2}}. \tag{5.14}$$

In this case the t^1 flow and the master symmetry reduce to

$$\partial_{t^1} u_{m,n} = u_{m,n}\Delta_m\left(\frac{u_{m-1,n+1}}{\Gamma}\right), \quad \partial_\tau u_{m,n} = u_{m,n}\Delta_m\left(\frac{m u_{m-1,n+1}}{\Gamma}\right),$$

where $\Gamma = a(u_{m,n} + u_{m-1,n+1}) + u_{m,n}u_{m-1,n}u_{m,n+1}u_{m-1,n+1}$, with $u_{m,n} = u^{(1)}_{m,n}$ and $\partial_\tau a = 1$.

To construct the s^1 flow, we solve the equations for $W^{(i)}_{m,n}$ to obtain

$$\begin{aligned}
W^{(1)}_{m,n} &= \mathrm{diag}\left(\frac{1}{v^{(0)}_{m,n}v^{(0)}_{m,n+1}}, \frac{1}{v^{(0)}_{m,n-1}v^{(0)}_{m,n-2}}, \frac{1}{v^{(0)}_{m,n}v^{(0)}_{m,n-1}}\right)\Omega^2,\\
W^{(2)}_{m,n} &= \mathrm{diag}\left(\frac{1}{v^{(0)}_{m,n}}, 0, \frac{1}{v^{(0)}_{m,n-1}}\right)\Omega,
\end{aligned}$$

to give

$$\partial_{s^1} v^{(0)}_{m,n} = \frac{1}{v^{(0)}_{m,n-1}v^{(0)}_{m,n-2}} - \frac{1}{v^{(0)}_{m,n+2}v^{(0)}_{m,n+1}}. \tag{5.15}$$

We can now take our solution for W and consider the compatibility (5.12). The off-diagonal terms vanish identically as a consequence of the defining constraints (for $v^{(0)}_{m,n}$, $u^{(0)}_{m,n}$, etc) and the difference equation (5.14). The s^1 evolution for $u^{(1)}_{m,n}$ (given by one of the components of (5.12)) is

$$\partial_{s^1} u^{(1)}_{m,n} = w^{12}_{m,n} - w^{11}_{m+1,n} = (\mathcal{S}_n - \mathcal{S}_m)\left(\frac{1}{v^{(0)}_{m,n-1} v^{(0)}_{m,n-2}}\right),$$

which can be written in terms of $u_{m,n} = u^{(1)}_{m,n}$:

$$\partial_{s^1} u_{m,n} = u_{m,n+1} u^2_{m,n} u_{m,n-1} \Delta_n \left(\frac{1}{\Gamma_{m,n}\Gamma_{m,n+1}}\right), \tag{5.16}$$

where $\Gamma_{m,n} = a - u_{m,n}u_{m,n-1}u_{m,n-2}$.

Example 5.5 (The Case of General N) Example 5.4 can be extended to general N, reducing again to an equation for the single function $u^{(1)}_{m,n}$, with

$$u^{(0)}_{m,n} = \frac{a}{\prod_{i=0}^{N-2} u^{(1)}_{m,n+i}}, \quad u^{(i)}_{m,n} = u^{(1)}_{m,n+i-1}, \ \ i = 2, \dots, N-1, \quad v^{(0)}_{m,n} = \frac{a - \prod_{i=0}^{N-1} u^{(1)}_{m,n+i}}{\prod_{i=0}^{N-2} u^{(1)}_{m,n+i}}.$$

Writing $u_{m,n} = u^{(1)}_{m,n}$, we obtain the difference equation

$$u_{m,n} + \frac{a}{\prod_{i=0}^{N-2} u_{m+1,n+i}} = u_{m+1,n+N-1} + \frac{a}{\prod_{i=1}^{N-1} u_{m,n+i}}. \tag{5.17a}$$

The $u^{(1)}_{m,n}$ component of the t^1 flow can then be written in the form

$$\partial_{t^1} u_{m,n} = u_{m,n}\Delta_m \left(\frac{\prod_{j=1}^{N-2} u_{m-1,n+j}}{Y_{m-1,n}}\right), \tag{5.17b}$$

where

$$Y_{m,n} = a \sum_{j=1}^{N-1}\left(\prod_{p=0}^{N-j-2} u_{m+1,n+p} \prod_{s=N-j}^{N-2} u_{m,n+s}\right) + \prod_{j=0}^{N-2} u_{m,n+j}u_{m+1,n+j}.$$

In this formula, we use the convention that $\prod_{k=a(j)}^{b(j)} P_k = 1$, whenever $a(j) > b(j)$.

The first $s-$flow (5.13a) is now

$$\partial_{s^1} v^{(0)}_{m,n} = \prod_{j=1}^{N-1} \frac{1}{v^{(0)}_{m,n-j}} - \prod_{j=1}^{N-1} \frac{1}{v^{(0)}_{m,n+j}}, \tag{5.17c}$$

while the $u^{(1)}_{m,n}$ (5.13b), with $u^{(1)}_{m,n} = u_{m,n}$, takes the form

$$\partial_{s^1} u_{m,n} = \prod_{j=0}^{N-2} \frac{1}{v^{(0)}_{m,n-j}} - \prod_{j=1}^{N-1} \frac{1}{v^{(0)}_{m+1,n-j}} = P_N \Delta_n \left(\prod_{j=0}^{N-2} \frac{1}{\Gamma_{m,n+j}}\right), \tag{5.17d}$$

where

$$P_N = \prod_{i=0}^{N-2} \rho_i, \quad \text{with} \quad \rho_i = \prod_{j=-i}^{i} u_{m,n+j} \quad \text{and} \quad \Gamma_{m,n} = a - \prod_{j=0}^{N-1} u_{m,n-j}.$$

5.2.2 The Modified Bogoyavlenskii Lattice Equation

As a consequence of Equation (5.17c), the variable $v = \frac{1}{v^{(0)}}$ satisfies one of the modified Bogoyavlenskii lattice equations [3]:

$$\partial_s v_{m,n} \;=\; v_{m,n}^2 \left(\prod_{j=1}^{N-1} v_{m,n+j} \;-\; \prod_{j=1}^{N-1} v_{m,n-j} \right), \tag{5.18}$$

so we can state the following:

Proposition 5.6 *A Lax pair for the modified Bogoyavlenskii lattice equation (5.18) is given by*

$$\Psi_{m,n+1} \;=\; \left(\frac{1}{v_{m,n}} W^{(N)} \;+\; \lambda\,\Omega \right) \Psi_{m,n}\,, \quad \partial_s \Psi_{m,n} \;=\; - \left(\sum_{i=1}^{N} \frac{1}{\lambda^i}\, W^{(i)} \right) \Psi_{m,n}\,, \tag{5.19a}$$

in which the $N \times N$ matrices $W^{(j)}$ are given by

$$\begin{aligned} W^{(j)} \;&=\; \operatorname{diag}\left(p_{m,n}^{(j)}, \mathbf{0}^{(j-1)}, p_{m,n+j-N}^{(j)}, p_{m,n+j-N+1}^{(j)}, \ldots, p_{m,n-1}^{(j)} \right) \Omega^{N-j}, \\ &\qquad\qquad j = 1, \ldots, N-1, \qquad (5.19\text{b}) \\ W^{(N)} \;&=\; \operatorname{diag}\left(1, 0, \cdots, 0\right), \qquad (5.19\text{c}) \end{aligned}$$

where functions $p_{m,n}^{(j)}$ are defined as

$$p_{m,n}^{(j)} \;:=\; \prod_{k=0}^{N-j-1} v_{m,n+k}\,, \tag{5.19d}$$

and symbol $\mathbf{0}^{(j-1)}$ in the definition of matrices $W^{(j)}$ stands for $(j-1)$ consecutive zero entries.

Furthermore, (5.18) is related to (5.17d) through the Miura transformation

$$v_{m,n} = \frac{\prod_{i=0}^{N-2} u_{m,n+i}}{a - \prod_{i=0}^{N-1} u_{m,n+i}}.$$

Corresponding to the difference equation (5.17a) for $u_{m,n}$, we can derive a difference equation for $v_{m,n}$, for low values of N. For $N = 2, 3$, it is given respectively by

$$(v_{m,n}v_{m,n+1} - v_{m+1,n}v_{m+1,n+1})^2 - a(v_{m,n} - v_{m+1,n+1})(v_{m+1,n} - v_{m,n+1}) = 0$$

and

$$\begin{aligned} &(v_{m,n}v_{m,n+1}v_{m,n+2} - v_{m+1,n}v_{m+1,n+1}v_{m+1,n+2})^3 + \\ &a(v_{m,n} - v_{m+1,n+2})(v_{m+1,n}v_{m+1,n+1} - v_{m,n+1}v_{m,n+2})(v_{m,n}v_{m,n+1}(v_{m,n+2} - v_{m+1,n+1}) \\ &+ v_{m+1,n+1}v_{m+1,n+2}(v_{m,n+1} - v_{m+1,n})) = 0. \end{aligned}$$

The case $N = 2$ can be found in the classification of [2] ($H3^*$ with $q = 0$), whilst the case of $N = 3$ is new.

6 Nonlocal Symmetries and the Relation to $2D$ Toda Systems

In the previous section, we derived (generically) *local* continuous symmetries for the discrete system (2.5), as described in Proposition 4.2. In that case, $S_{m,n}$ of the Lax pair (3.1) had specific forms given by (4.5c) and (4.5d).

In the current section we present two different solutions for $S_{m,n}$, one linear in λ and the other proportional to λ^{-1}, which give *nonlocal* symmetries of the entire class of fully discrete systems (2.5) (only the first of these is nontrivial for the *degenerate case*).

The Lax pair given in [6] for the $2D$ Toda lattice is generalized to (6.1) yielding nonlocal symmetries for our fully discrete system. In this way, we associate a $2D$ Toda lattice with each equation in the entire class discussed in this paper. In the generic (non-degenerate and non-reduced) case, the nonlocal symmetries are just the components of the corresponding Bäcklund transformation and the discrete system the corresponding nonlinear superposition formula.

In the case of reduced equations, such as (6.14), this correspondence is not so straight forward. However, in this case, we use our nonlocal symmetries to derive Schief's Bäcklund transformation [13] for the Tzitzeica equation. It is still not clear what role is being played by the discrete equation (6.14).

6.1 Nonlocal Symmetries

We consider the compatibility of the discrete Lax equations (2.1a) and (2.1b) with the continuous evolutions

$$\partial_x \Psi_{m,n} = \left(A_{m,n} + \lambda \Omega^{\ell_1 - k_1}\right) \Psi_{m,n}\,, \tag{6.1a}$$

$$\partial_y \Psi_{m,n} = \frac{1}{\lambda} B_{m,n} \Psi_{m,n}\,. \tag{6.1b}$$

For compatibility with either (2.1a) or (2.1b), A and B must have the following level structure

$$A_{m,n} = \mathrm{diag}(a^{(0)}_{m,n}, a^{(1)}_{m,n}, \dots, a^{(N-1)}_{m,n}),$$
$$B_{m,n} = \mathrm{diag}(b^{(0)}_{m,n}, b^{(1)}_{m,n}, \dots, b^{(N-1)}_{m,n})\Omega^{k_1-\ell_1} = \mathrm{diag}(b^{(0)}_{m,n}, b^{(1)}_{m,n}, \dots, b^{(N-1)}_{m,n})\Omega^{k_2-\ell_2}.$$

The calculations for the two cases are identical, just requiring the change of labels, $(u, k_1, \ell_1) \mapsto (v, k_2, \ell_2)$. The condition (2.2) allows them to be *simultaneously* compatible. The results can be summarised as follows:

Proposition 6.1 *The compatibility of (6.1a) with (2.1a) and (2.1b) leads to the following equations*

$$\partial_x u^{(i)}_{m,n} = u^{(i)}_{m,n}\left(a^{(i)}_{m+1,n} - a^{(i+k_1)}_{m,n}\right), \tag{6.2a}$$

$$\partial_x v^{(i)}_{m,n} = v^{(i)}_{m,n}\left(a^{(i)}_{m,n+1} - a^{(i+k_2)}_{m,n}\right), \tag{6.2b}$$

where

$$a^{(i)}_{m+1,n} - a^{(i+\ell_1)}_{m,n} = u^{(i)}_{m,n} - u^{(i+\ell_1-k_1)}_{m,n}, \tag{6.3a}$$

$$a^{(i)}_{m,n+1} - a^{(i+\ell_2)}_{m,n} = v^{(i)}_{m,n} - v^{(i+\ell_2-k_2)}_{m,n} \tag{6.3b}$$

whilst the compatibility of (6.1b) with (2.1a) and (2.1b) leads to

$$\partial_y u^{(i)}_{m,n} = b^{(i)}_{m+1,n} - b^{(i+\ell_1)}_{m,n}, \tag{6.4a}$$

$$\partial_y v^{(i)}_{m,n} = b^{(i)}_{m,n+1} - b^{(i+\ell_2)}_{m,n}, \tag{6.4b}$$

where

$$b^{(i)}_{m+1,n}\,u^{(i+k_1-\ell_1)}_{m,n} = u^{(i)}_{m,n}\,b^{(i+k_1)}_{m,n}, \tag{6.5a}$$

$$b^{(i)}_{m,n+1}\,v^{(i+k_2-\ell_2)}_{m,n} = v^{(i)}_{m,n}\,b^{(i+k_2)}_{m,n}, \tag{6.5b}$$

Equations (6.2) and (6.4) define nonlocal symmetries for our general discrete system (2.5).

The $y-$flow does not always exist for the degenerate cases of the classification given in Section 2.1.

6.1.1 Quotient Potential Form

Using the quotient potentials (2.8a), equations (6.2) integrate to give $a^{(i)}_{m,n} = \partial_x(\log\phi^{(i)}_{m,n})$, whilst (6.3) lead to

$$\partial_x \log\left(\frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+\ell_1)}_{m,n}}\right) = \alpha\left(\frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}} - \frac{\phi^{(i+\ell_1-k_1)}_{m+1,n}}{\phi^{(i+\ell_1)}_{m,n}}\right), \tag{6.6a}$$

$$\partial_x \log\left(\frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+\ell_2)}_{m,n}}\right) = \beta\left(\frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}} - \frac{\phi^{(i+\ell_1-k_1)}_{m,n+1}}{\phi^{(i+\ell_2)}_{m,n}}\right), \tag{6.6b}$$

On the other hand, (6.5) lead to $b^{(i)}_{m,n} = \frac{\phi^{(i)}_{m,n}}{\phi^{(i+k_1-\ell_1)}_{m,n}}$, with (6.4) giving

$$\partial_y \log\left(\frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}}\right) = \frac{1}{\alpha}\left(\frac{\phi^{(i+k_1)}_{m,n}}{\phi^{(i+k_1-\ell_1)}_{m+1,n}} - \frac{\phi^{(i+\ell_1)}_{m,n}}{\phi^{(i)}_{m+1,n}}\right), \tag{6.7a}$$

$$\partial_y \log\left(\frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}}\right) = \frac{1}{\beta}\left(\frac{\phi^{(i+k_2)}_{m,n}}{\phi^{(i+k_2-\ell_2)}_{m,n+1}} - \frac{\phi^{(i+\ell_2)}_{m,n}}{\phi^{(i)}_{m,n+1}}\right). \tag{6.7b}$$

Equations (6.6) and (6.7) define nonlocal symmetries for the potential system (2.8b).

Example 6.2 (Modified KdV : $N = 2$, structure $(0,1;0,1)$) The x flow (see also [1]) takes the form

$$\partial_x \log(\phi_{m,n}\phi_{m+1,n}) = \alpha\left(\frac{\phi_{m+1,n}}{\phi_{m,n}} - \frac{\phi_{m,n}}{\phi_{m+1,n}}\right),$$

$$\partial_x \log(\phi_{m,n}\phi_{m,n+1}) = \beta\left(\frac{\phi_{m,n+1}}{\phi_{m,n}} - \frac{\phi_{m,n}}{\phi_{m,n+1}}\right).$$

The y flow is

$$\alpha\partial_y \log\left(\frac{\phi_{m+1,n}}{\phi_{m,n}}\right) = \phi_{m,n}\phi_{m+1,n} - \frac{1}{\phi_{m,n}\phi_{m+1,n}},$$

$$\beta\partial_y \log\left(\frac{\phi_{m,n+1}}{\phi_{m,n}}\right) = \phi_{m,n}\phi_{m,n+1} - \frac{1}{\phi_{m,n}\phi_{m,n+1}}.$$

6.1.2 Additive Potential Form

In the additive potential form (2.10a), equations (6.4) immediately integrate to give $b^{(i)}_{m,n} = \partial_y \chi^{(i)}_{m,n}$, whilst (6.5) lead to

$$\begin{aligned} \partial_y \chi^{(i)}_{m+1,n}(\chi^{(i+k_1-\ell_1)}_{m+1,n} - \chi^{(i+k_1)}_{m,n}) &= (\chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n})\partial_y \chi^{(i+k_1)}_{m,n}, & \text{(6.8a)} \\ \partial_y \chi^{(i)}_{m,n+1}(\chi^{(i+k_2-\ell_2)}_{m,n+1} - \chi^{(i+k_2)}_{m,n}) &= (\chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n})\partial_y \chi^{(i+k_2)}_{m,n}. & \text{(6.8b)} \end{aligned}$$

On the other hand, (6.3) lead to $a^{(i)}_{m,n} = \chi^{(i)}_{m,n} - \chi^{(i+\ell_1-k_1)}_{m,n}$, with (6.2) giving

$$\begin{aligned} \partial_x \left(\log\left(\chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}\right)\right) &= \chi^{(i)}_{m+1,n} - \chi^{(i+k_1)}_{m,n} - \left(\chi^{(i+\ell_1-k_1)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}\right), & \text{(6.9a)} \\ \partial_x \left(\log\left(\chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}\right)\right) &= \chi^{(i)}_{m,n+1} - \chi^{(i+k_2)}_{m,n} - \left(\chi^{(i+\ell_2-k_2)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}\right). & \text{(6.9b)} \end{aligned}$$

Equations (6.9) and (6.8) define nonlocal symmetries for the potential system (2.10b).

Example 6.3 (Potential KdV : $N = 2$, structure $(0,1;0,1)$) The x flows (see also [1]), after some manipulation, take the form

$$\partial_x (\chi_{m+1,n} + \chi_{m,n}) = (\chi_{m+1,n} - \chi_{m,n})^2 + \alpha^2 ,$$
$$\partial_x (\chi_{m,n+1} + \chi_{m,n}) = (\chi_{m,n+1} - \chi_{m,n})^2 + \beta^2.$$

The second function $\chi^{(1)}$ cannot be removed from the y flow. It remains in the formulae as a pseudo-potential ψ. Explicitly,

$$\partial_y \chi_{m+1,n} = \frac{(\chi_{m+1,n} - \psi_{m,n})^2}{\alpha^2} \partial_y \chi_{m,n} , \quad \partial_y \chi_{m,n+1} = \frac{(\chi_{m,n+1} - \psi_{m,n})^2}{\beta^2} \partial_y \chi_{m,n},$$

where $\psi_{m,n}$ is pseudo-potential such that

$$(\chi_{m+1,n} - \psi_{m,n})(\chi_{m,n} - \psi_{m+1,n}) = \alpha^2, \quad (\chi_{m,n+1} - \psi_{m,n})(\chi_{m,n} - \psi_{m,n+1}) = \beta^2.$$

In fact, ψ is another solution of H1 related to χ via the above Bäcklund transformation.

Example 6.4 (Schwarzian KdV : $N = 2$, structure $(1,0;1,0)$) The x flows, after some manipulation, take the form

$$\begin{aligned} \partial_x (\chi_{m+1,n} - \chi_{m,n}) &= (\chi_{m+1,n} - \chi_{m,n}) (\chi_{m+1,n} + \chi_{m,n} - \psi_{m+1,n} - \psi_{m,n}) , \\ \partial_x (\chi_{m,n+1} - \chi_{m,n}) &= (\chi_{m,n+1} - \chi_{m,n}) (\chi_{m,n+1} + \chi_{m,n} - \psi_{m,n+1} - \psi_{m,n}) , \end{aligned}$$

where the pseudo-potential ψ is determined by the relations

$$\psi_{m+1,n} - \psi_{m,n} = \frac{\alpha^2}{\chi_{m+1,n} - \chi_{m,n}} , \quad \psi_{m,n+1} - \psi_{m,n} = \frac{\beta^2}{\chi_{m,n+1} - \chi_{m,n}}.$$

The y flow (see also [1]) takes the form

$$(\partial_y \chi_{m+1,n}) (\partial_y \chi_{m,n}) = \frac{(\chi_{m+1,n} - \chi_{m,n})^2}{\alpha^2} , \quad (\partial_y \chi_{m,n+1}) (\partial_y \chi_{m,n}) = \frac{(\chi_{m,n+1} - \chi_{m,n})^2}{\beta^2}.$$

6.2 Associated $2D$ Toda Lattices

The compatibility condition of (6.1a) and (6.1b) leads to the equations

$$\partial_y a^{(i)}_{m,n} = b^{(i)}_{m,n} - b^{(i+\ell_1-k_1)}_{m,n}, \tag{6.10a}$$

$$\partial_x \log\left(b^{(i)}_{m,n}\right) = a^{(i)}_{m,n} - a^{(i+k_1-\ell_1)}_{m,n}. \tag{6.10b}$$

6.2.1 Quotient Potential Form

Here we have

$$a^{(i)}_{m,n} = \partial_x(\log\phi^{(i)}_{m,n}) \quad \text{and} \quad b^{(i)}_{m,n} = \frac{\phi^{(i)}_{m,n}}{\phi^{(i+k_1-\ell_1)}_{m,n}},$$

so (6.10b) is identically satisfied, whilst (6.10a) leads to

$$\partial_x\partial_y\theta^{(i)}_{m,n} = \exp(\theta^{(i)}_{m,n} - \theta^{(i+k_1-\ell_1)}_{m,n}) - \exp(\theta^{(i+\ell_1-k_1)}_{m,n} - \theta^{(i)}_{m,n}), \tag{6.11}$$

where $\phi^{(i)}_{m,n} = \exp(\theta^{(i)}_{m,n})$. These are the *Toda lattice equations* for $\theta^{(i)}_{m,n}$.

The nonlocal symmetries (6.6a) and (6.7a) then take the form of the *Bäcklund transformation*

$$\begin{aligned}
\partial_x\left(\theta^{(i)}_{m+1,n} - \theta^{(i+\ell_1)}_{m,n}\right) &= \alpha\left(\exp(\theta^{(i)}_{m+1,n} - \theta^{(i+k_1)}_{m,n}) - \exp(\theta^{(i+\ell_1-k_1)}_{m+1,n} - \theta^{(i+\ell_1)}_{m,n})\right), \\
\partial_y\left(\theta^{(i)}_{m+1,n} - \theta^{(i+k_1)}_{m,n}\right) &= \frac{1}{\alpha}\left(\exp(\theta^{(i+k_1)}_{m,n} - \theta^{(i+k_1-\ell_1)}_{m+1,n}) - \exp(\theta^{(i+\ell_1)}_{m,n} - \theta^{(i)}_{m+1,n})\right),
\end{aligned} \tag{6.12}$$

whilst the nonlocal symmetries (6.6b) and (6.7b) take the form of the *Bäcklund transformation*

$$\begin{aligned}
\partial_x\left(\theta^{(i)}_{m,n+1} - \theta^{(i+\ell_2)}_{m,n}\right) &= \beta\left(\exp(\theta^{(i)}_{m,n+1} - \theta^{(i+k_2)}_{m,n}) - \exp(\theta^{(i+\ell_2-k_2)}_{m,n+1} - \theta^{(i+\ell_2)}_{m,n})\right), \\
\partial_y\left(\theta^{(i)}_{m,n+1} - \theta^{(i+k_2)}_{m,n}\right) &= \frac{1}{\beta}\left(\exp(\theta^{(i+k_2)}_{m,n} - \theta^{(i+k_2-\ell_2)}_{m,n+1}) - \exp(\theta^{(i+\ell_2)}_{m,n} - \theta^{(i)}_{m,n+1})\right).
\end{aligned} \tag{6.13}$$

Remark 6.5 ($k_2 - \ell_2 \equiv k_1 - \ell_1 \ (\mathrm{mod} N)$) *Notice that only the combination $k_1 - \ell_1$ appears in the Toda lattice equations. The relation (2.2) therefore guarantees that both of these Bäcklund transformations lead to the same Toda lattice equations. Furthermore, different choices of k_i, ℓ_i, subject only to their difference $k_i - \ell_i$ remaining constant, lead to different forms of the Bäcklund transformation for the same Toda Lattice equation.*

Remark 6.6 (Sum of the Bäcklund Equations) *Summing the equations of (6.12), we find*

$$\partial_x\left(\sum_{i=0}^{N-1}\theta^{(i)}_{m+1,n}\right) = \partial_x\left(\sum_{i=0}^{N-1}\theta^{(i+\ell_1)}_{m,n}\right) \quad \textit{and} \quad \partial_y\left(\sum_{i=0}^{N-1}\theta^{(i)}_{m+1,n}\right) = \partial_y\left(\sum_{i=0}^{N-1}\theta^{(i+k_1)}_{m,n}\right),$$

and similarly for (6.13). In fact, as a consequence of (2.9b), we have $\sum_{i=0}^{N-1}\theta^{(i)}_{m,n} = 0$.

As a consequence, only $N-1$ of the Bäcklund equations are independent, for each of the four sets of equations in (6.12) and (6.13).

Example 6.7 (The Tzitzeica Reduction) The choice $N = 3$, with $k_i = 1, \ell_i = 2$ corresponds to the level structure $(1,2;1,2)$ and equation (4.16), which admits the reduction (first introduced in [10])

$$\phi_{m,n}\phi_{m+1,n+1}\left(\phi_{m+1,n} + \phi_{m,n+1}\right) = 2, \tag{6.14}$$

where

$$\phi^{(0)}_{m,n} = \phi^{(1)}_{m,n} = \frac{1}{\phi_{m,n}}, \quad \beta = -\alpha. \tag{6.15}$$

In the present context, it can be seen that equations (6.11) admit the reduction $\theta^0_{m,n} = -\theta^1_{m,n} = \theta_{m,n}$, $\theta^2_{m,n} = 0$, giving rise to the single component equation

$$\partial_x\partial_y\theta_{m,n} = e^{\theta_{m,n}} - e^{-2\theta_{m,n}}, \tag{6.16}$$

known as the Tzitzeica equation. However, the reduction of the Bäcklund equations to this case is not so simple. Setting $\theta^{(2)}_{m+1,n} = -\theta^{(0)}_{m+1,n} - \theta^{(1)}_{m+1,n}$, the four independent equations of (6.12) are

$$\begin{aligned}
\partial_x\theta^{(0)}_{m+1,n} &= \alpha\left(e^{\theta^{(0)}_{m+1,n}+\theta_{m,n}} - e^{\theta^{(1)}_{m+1,n}}\right), && \text{(6.17a)}\\
\partial_x(\theta^{(1)}_{m+1,n} - \theta_{m,n}) &= \alpha\left(e^{\theta^{(1)}_{m+1,n}} - e^{-\theta^{(0)}_{m+1,n}-\theta^{(1)}_{m+1,n}-\theta_{m,n}}\right), && \text{(6.17b)}\\
\partial_y(\theta^{(0)}_{m+1,n} + \theta_{m,n}) &= \frac{1}{\alpha}\left(e^{\theta^{(0)}_{m+1,n}+\theta^{(1)}_{m+1,n}-\theta_{m,n}} - e^{-\theta^{(0)}_{m+1,n}}\right), && \text{(6.17c)}\\
\partial_y\theta^{(1)}_{m+1,n} &= \frac{1}{\alpha}\left(e^{-\theta^{(0)}_{m+1,n}} - e^{\theta_{m,n}-\theta^{(1)}_{m+1,n}}\right). && \text{(6.17d)}
\end{aligned}$$

Equations (6.17a) and (6.17d) imply

$$\partial_x\left(\frac{1}{\alpha}e^{-\theta^{(0)}_{m+1,n}}\right) = \partial_y\left(\alpha e^{\theta^{(1)}_{m+1,n}}\right). \tag{6.18}$$

One option is to set $\theta^{(1)}_{m+1,n} = -\theta^{(0)}_{m+1,n}$, but this immediately implies

$$(\partial_x - \alpha^2\partial_y)\theta^{(0)}_{m+1,n} = 0 \quad \text{and} \quad (\partial_x - \alpha^2\partial_y)\theta_{m,n} = 0,$$

with each function satisfying (6.16), so they represent *travelling wave solutions*, related by

$$\partial_z\theta^{(0)}_{m+1,n} = \frac{1}{\alpha}\left(e^{\theta^{(0)}_{m+1,n}+\theta_{m,n}} - e^{-\theta^{(0)}_{m+1,n}}\right), \quad \partial_z\theta_{m,n} = \frac{1}{\alpha}\left(e^{\theta^{(0)}_{m+1,n}+\theta_{m,n}} - e^{-\theta_{m,n}}\right),$$

where each of the two functions depends only upon $z = \alpha^2 x + y$.

To avoid this degeneration, we use (6.18) to define the *potential function* $w(x,y)$, satisfying

$$\frac{1}{\alpha}e^{-\theta^{(0)}_{m+1,n}} = -\,\partial_y\log(w), \quad \alpha e^{\theta^{(1)}_{m+1,n}} = -\,\partial_x\log(w).$$

The remaining equations of (6.17) then imply

$$w_{xx} = \theta_x w_x - \alpha^3 e^{-\theta} w_y, \quad w_{xy} = e^{\theta} w, \quad w_{yy} = \theta_y w_y - \frac{e^{-\theta} w_x}{\alpha^3},$$

where $\theta = \theta_{m,n}$ and x and y suffices denote partial derivatives. These are the equations derived by Schief in [12, 13] for the Bäcklund transformation of the Tzitzeica equation.

6.2.2 Additive Potential Form

Here we have

$$a_{m,n}^{(i)} = \chi_{m,n}^{(i)} - \chi_{m,n}^{(i+\ell_1-k_1)} \quad \text{and} \quad b_{m,n}^{(i)} = \partial_y \chi_{m,n}^{(i)},$$

so (6.10a) is identically satisfied, whilst (6.10b) leads to

$$\partial_x \log\left(\partial_y\, \chi_{m,n}^{(i)}\right) = 2\chi_{m,n}^{(i)} - \chi_{m,n}^{(i+k_1-\ell_1)} - \chi_{m,n}^{(i+\ell_1-k_1)}.$$

Differentiating this with respect to y and defining $\rho_{m,n}^{(i)} = \log(\partial_y\, \chi_{m,n}^{(i)})$, we obtain an alternative form of the Toda equations

$$\partial_x \partial_y \rho_{m,n}^{(i)} = 2\exp(\rho_{m,n}^{(i)}) - \exp(\rho_{m,n}^{(i+k_1-\ell_1)}) - \exp(\rho_{m,n}^{(i+\ell_1-k_1)}). \tag{6.19}$$

6.3 Degenerate Case and Nonlocal Symmetries

We do not have general formulae in this case, so just present an explicit example.

Example 6.8 (Hirota's KdV Equation) For Hirota's KdV equation (5.4), a nonlocal symmetry is generated by

$$\partial_x \log\left(u_{m,n}\right) \;=\; -2a_{m,n}^{(0)} + u_{m,n} \;-\; \frac{a}{u_{m,n}}\,, \tag{6.20a}$$

where

$$a_{m+1,n}^{(0)} = u_{m,n} \;-\; \frac{a}{u_{m,n}} \;-\; a_{m,n}^{(0)}\,, \quad a_{m,n+1}^{(0)} \;=\; u_{m,n} \;-\; \frac{a}{u_{m,n+1}} \;-\; a_{m,n}^{(0)}\,. \tag{6.20b}$$

7 Conclusions & discussion

In this paper we considered the continuous isospectral deformations of our discrete Lax matrices (2.1), so as to construct differential-difference equations which we interpret as symmetries of the discrete integrable systems presented in [7].

For the *generic coprime* case we gave a complete description of isospectral deformations of L (or M), corresponding to hierarchies of *autonomous* differential-difference equations. Furthermore, we presented a *non-autonomous flow* which played the role of a *master symmetry*, which was used to generate the autonomous hierarchy. When we ask for compatibility between these continuous flows and *both L and M*, then these differential-difference equations are interpreted as symmetries of the fully discrete system (2.5). This is another manifestation of the integrability of the fully discrete system.

Once again, the degenerate case is less systematic. Whilst the compatibility with L is still guaranteed under this reduction, the compatibility with M depends upon the specific values of k_2 and ℓ_2. Whilst the $t-$flows continue to exist, the $s-$flows may trivialise or become *nonlocal*. As a result, it is evident that our general construction of the corresponding master symmetry (σ flow) is no longer valid. In this paper we considered two extreme cases of degeneracy. In one case, with only one <u>zero</u> components ($v^{(N-1)} = 0$), we could still solve our equations (5.3). On the other hand, with only one <u>non-zero</u> component ($v^{(0)} \neq 0$), we introduced a more general time evolution (5.11) and derived some interesting single component difference equations, whose symmetries are Miura related to a Bogoyavlenskii lattice equation. This leaves a plethora of other degenerate cases, which we believe to be a rich source of low dimensional reductions. We comment that another Bogoyavlenskii lattice arises in the context of symmetries of the self-dual generic system for $N = 3$ [8]. For $N > 3$, the corresponding systems give rise to multi-component generalisations of the Bogoyavlenskii lattice.

In Section 6, we looked at a specific class of *nonlocal symmetries* and again were able to give a complete analysis in the *generic coprime* case. We gave two such symmetries (labelled $x-$ and $y-$flows), which again had natural representations in both the *quotient* and *additive* potential forms, giving simple forms for nonlocal symmetries of the corresponding potential forms of the fully discrete system. These two forms naturally gave rise to two forms of the Toda lattice equations. The quotient form immediately gives the standard form of the Toda lattice equations, with the nonlocal symmetries taking the form of components of the corresponding Bäcklund transformation. In fact, for *each* Toda lattice equation, we obtain a sequence of different Bäcklund transformations. Thus, in the three component case with $\ell_i - k_i = 1$, we have three such transformations. These can be reduced to the Bäcklund transformation for the Tzitzeica equation [12, 13], but only one case possesses a reduced discrete system (equation (6.14)).

There are a number of open questions. We currently have no *proof* that our master symmetries generate *commuting hierarchies*, but have not found a counter-example in our examples. We have found several *reductions* to lower dimensional systems (such as (6.14)) but have no systematic way of analysing these. Master symmetries in the degenerate case can be obtained through non-isospectral deformations, but extending beyond the simplest examples leads to non-localities. The connection with well known reduced PDEs, such as the Sawada-Kotera and Hirota-Satsuma equations is also not clear. In this paper, we restricted our Lax pairs to be *linear* in λ. Similar Lax pairs, polynomial in λ would be interesting to consider.

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