Positional Non-Cooperative Equilibrium

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Abstract This paper presents and analyses a game theoretic model for resource allocation, where agents are status-seeking and consuming positional goods. We propose a unified framework to study the competition for resources where agents' preferences are not necessarily ordered according to the absolute amount of goods they consume, but may depend on the consumption of others as well as on individual valuation of the goods at stake. Our model explicits the relation between absolute good distribution, individual evaluation and the level of consumption adopted by the opponents; such relation has the form of a status function. We show that given a certain set of properties, there exists only one possible status function. The competition mechanism implemented to maximise one own's status is central in this work. As a result of the mathematical formulation, we show that the standard utility-maximisation paradigm emerges as a special case (non-positional competition). We then define a new class of games where the individual evaluations are negotiable and serve only the purpose of maximising one own's status.

Keywords positional good, non-cooperative game, Nash equilibrium, correlated strategies, status-seeking agents

1. Introduction

In traditional economics model, individual utility depends only on the absolute consumption of goods [1]; social interactions, comparisons in the level of consumption and social ranking are of no relevance for resource distribution. Though the importance of relative allocation in the economic behaviour can be traced back to the works of [2] and [3], it is not before the last decade that researchers in both economics and social sciences have tried to find alternative frameworks to describe this type of dynamics.

There is a broad literature of empirical evidence, initiated by [4] with his famous paradox, [5], [6], [7] and many more, which confirms that people evaluate their consumption in relation to that of others. As suggested by [8], in order to be able to understand and model the competition in such a complex scenario, we must reject the hypothesis held by standard economic models for which the ultimate goal of desire is just to consume goods.

In fact if we recognise that desire can be driven by non-material sources such as *human desire for distinction*, psychological factors or social practices, then the notion of revealed preference, based on a naïve idea of utility [9], might not be adequate to explain a large range of human behaviours.

The term *positional good* has been coined by [10]. In his original formulation, positional goods are those goods that are consumed only for demonstration purposes and whose role is only to determine the relative standing of an individual in the community. In contrast, a non-positional good has only consumption value and cannot be used to gain status in the society.

The main differences with public or private goods rely on the fact that consumption of a positional good generates positive externalities for the consumer and negative externalities for at least one other individual; hence, in this light, a positional good can be positively or negatively consumed. As remarked in [8], a positional good is a double rival and double excludable good in consumption; in fact, agents are rival on the positive and negative consumption (double rivalry) and, agents must be able to exclude others from the returns of positive consumption and be excluded by the returns of negative consumption (double excludability).

Eventually, if there is a party consuming a positive amount of positional good, there must be a counterpart who is consuming a negative amount of the same good. Based on this dynamic, which is peculiar only to positional goods, the formulation of the problem of positional resource allocation as a zero-sum-game is justified [11].

Pagano's contribution in [11], as Hirsch's original formulation, is based on the assumption that the positive externalities produced by the consumption of a positional good are balanced by negative externalities of the same magnitude, resulting in a zero-sum game.

But this is not necessarily true for the negative consumption of a good as perceived by an individual could differ in intensity from the positive consumption that has generated it. In the first part of this work, we try to address this problem by extending the notion of *positionality* from its boolean formulation (positional/non positional) to a continuous range of intermediate values from which Hirsch's categories emerge and, therefore, Pagano's model results as a special case.

As pointed out in [12], the concept of positionality is fundamental to the idea of *status* and hence, every economic model that aims to explain the dynamic of status-seeking agents, must contain a description of the mechanism that controls positional competition.

Indeed in its most abstract form, rational choice theory is general enough to incorporate any assumption about the nature of preferences, including the assumptions about the objects over which prederences are defined [12]. This means that utilities are general enough to represent positional payoffs. However, in our view, an exhaustive model should contain an explicit formulation of the salient characteristics of status and not just a black-boxed numerical representation of well-being. In [12] the authors identify three basic features of social status: positionality, desiderability and non-tradability. They intend positionality as the relation of one's consumption with that of others, desiderability as the agent's attitude towards the acquisition of social status and non-tradability as the absence of a market for status that is an exogenous conferment of society.

In this work we propose a *status function* which we prove to be unique in respect to a set of particular properties as the basis of positional competition. Such a function is an explicit relation between absolute and relative consumption and incorporates the idea of different levels of intensity for the externalities produced by consumption. We identify four main properties plus two additional features that a status function should possess. This allows us to completely describe the equilibrium strategies for the allocation of positional goods in game theoretic terms. We employ the status function to calculate the consumption of positional goods that jointly maximise individuals' statuses.

From a pure game theoretic point of view we also give the solutions to approximated non-cooperative games for positional goods, exploiting the conditions of continuity of the status function. After proving the existence of such approximations, we provide an extensive study on diverse concepts of equilibrium (i.e. Nash, correlated and Bayes-Nash) and remark on the relation with the non-approximated formulation to the same allocation problem.

On the basis of a number of computer simulations, we show that the aggregate distribution of goods assigned to the players when an exogenous value for the positionality index is provided, is Pareto-superior in consumption to the distribution in presence of unrelated levels of positionality. In particular, in the case of correlated equilibria we obtain solutions which are Pareto-optimal. In our view these results confirm the ideas in [12] and [9] that social regulations could prevent positional competition from generating inefficient solutions.

In conclusion of the game theoretic exposition, we examine a different type of dynamic that emerges when each individual's positionality index has the only objective of conditioning the consumption at equilibrium. We define the resulting games as *negotiable* positionality index games, for each player can opportunistically select the positionality index that increases his level of absolute consumption.

An important aspect of the status competition is represented by the possibility of aggregating in *coalitions* with the goal of increasing the decisional power of the participants in a coalition over a distribution of resources. As remarked in [12] and [9] there is a loss of efficiency in the competitive consumption of positional goods when compared to a cooperative solution; therefore, in general, social regulations and policies are required to prevent the over-consumption of positional resources.

Again in [8], the author remarks on the validity of the principle of the *invisible hand*, which is valid only in the special case where each individual's rewards are completely independent from the choices made by others. We provide a mechanism for the rational adjustment of the levels of consumption that increases the efficiency of the competitive solution. Such a mechanism allows the formation of coalitions after the competitive process has ended. We show that the limit to the presence of stable coalitions (those where no individual can increase his status by free riding) is the Nash solution to the bargaining problem.

The remainder of this paper is structured as follows

Section 2 provides the definition of positionality index and introduces the model of absolute consumption.

Sections 3 and 4 investigate the properties of a status function and its mathematical formulation.

Section 5 defines the conditions for the existence of an equilibrium point in non-cooperative games for positional goods.

In section 6 we propose the equilibrium approximations for the game discussed in the preceeding section and present some results and their significance in relation to the non-approximated game; in particular section 6.2 gives a formulation of the problem of positional allocation of goods in the presence of incomplete information.

In section 7 we relax the property of non-tradability [12] to introduce the novel concept of equilibrium in negotiable positionality index games when players' positional engagement is not *a-priori* determined.

Sections 8 and 9 describe the cooperative mechanism to adjust the positional distribution and, eventually, in the final section we discuss the concepts we developed and draw the conclusions to our work.

2. Positionality Index

According to [2] and [10], positionality is a binary feature, for a good is either used to signal one's own social status, or it is a consumption good. In this work we utilise the extension of this concept to a continuous range of degrees of positionality, representing the diverse intensity in the generated externalities.

We assume that, for any good, there exists a relation between its power of signalling one's status and the intensity of generated externalities. Thus, in this perspective every good is positional, it is the intensity of its signalling capability, and hence of the level of consequent externalities, that varies.

The spectrum of such magnitude spans from *public* to *private* to *positional*, with an infinite scale of intermediate variations. To represent the positional component we introduce a *positionality index* (π), formally a real number, which measures and orders goods according to the signalling power (generated externalities) perceived (produced) by the agent who consumes it.

By definition, the consumption of a good with $\pi = 0$ produces positive externalities for other individuals and thus it is a model of public good; private goods are associated with $\pi = 1$ for the consumption is not used to signal one's own social status. Eventually $\pi > \bar{\pi}$ are positional goods in the sense of Hirsch. Goods which are associated to non-extreme π have to be considered decisive in defining one's own status according to the natural ordering of $\mathbb{R}$.

Table 1 summarises the different meanings of the positionality index.

In this work we consider the positional component as the exponent of the absolute consumption: $q_i(\sigma)^\pi$, with $\sigma = (\sigma_i, \sigma_{-i}) \in \Sigma$ that represent a point in the joint mixed strategy set and with $q_i(\sigma)$ concave in σ_i.

The extension of the concept of positionality introduces the necessity of a more general framework for the analysis of multi player competition for goods; in fact, it is not necessarily true that the distribution of a positional good shall result in a zero-sum game [11]. There may exist goods which are more socially or naturally abundant that, however, are utilised to manifest social status.

Another assumption is to consider the social status produced by the consumption of zero units of a public good, equivalent to the consumption of one unit of any other type goods. Though this assumption may not be completely realistic, for it appears to be better for a status-seeking individual to possess no units of a public good rather than one unit of a good with high positionality, it helps us solve mathematical irregularities. In this work we consider $q_i(\sigma) > 1 \in \mathbb{R}$.

With this formulation, we have a categorisation of goods according to their positionality and a clear distinction between the absolute level of consumption and the positional value of such consumption. We need now to explicitly include the relative distribution of goods in order to produce an expressive economic model for status-seeking competition in accordance to [12]. For these reasons we introduce the status function.

3. Status function

In our model every non-private good, $\pi \neq 1$, can be replaced by a fictitious good without positionality $\pi = 1$. Thus competing for positional goods results in a competition for private goods where the distribution of such goods are modified according to the positionality index and the relative consumption. The transformation from a game of positional consumption to the corresponding competition for private goods is realised by means of a *status function*: $\delta_i(\sigma, \pi)$.

3.1. Properties

There are some fundamental properties that a status function must satisfy in order to express the dynamics of positional competition. The following set extends the list provided in [12]:

1. **Invariance**: Individuals' absolute consumption of non positional goods are invariants under the status transformation. $\forall \sigma \in \Sigma$ if $\pi = 1$ then $\delta_i(\sigma, 1) = q_i(\sigma)$
2. **Exclusivity**: Individual's status for high positional goods increases as the distribution gets more exclusive. $\exists \sigma^0, \sigma^1 \in \Sigma$ such that $\sum_{j \in N} q_j(\sigma^0) = \sum_{j \in N} q_j(\sigma^1)$ and $q_i(\sigma^0) = q_i(\sigma^1), \forall i \in N$, then $\delta_i(\sigma^0, \pi) \geq \delta_i(\sigma^1, \pi)$ if and only if $\sum_{j \in N} q_j(\sigma^0)^\pi \leq \sum_{j \in N} q_j(\sigma^1)^\pi$
3. **Simmetry**: If there is no difference in the relative distribution of a good for all π, then individuals' absolute consumption is unchanged after the status transformation. $\exists \sigma \in \Sigma$ such that $q_i(\sigma) = q_k(\sigma)$ with $i \neq k$, $\forall i, k \in N$, then $\delta_i(\sigma, \pi) = \delta_k(\sigma, \pi)$
4. **Efficiency**: In each state the sum of individual's distribution is equal to the total amount of available good. $\forall \sigma \in \Sigma \sum_{i \in N} \delta_i(\sigma, \pi) = \sum_{i \in N} q_i(\sigma)$

There exists no codified method to evaluate the positionality index, for it may depend on subjective evaluations or traditions which could be difficult to measure [8]. The scope of this work is to define a coherent model to analyse the dynamics of competition for Veblen goods for which a positionality measure is given.

3.2. Mathematical formulation

Duesenberry's hypothesis of relative income and Pollack's model [13] are the simplest way to incorporate relative preferences into a utility function. In this model the innovative element is the introduction of the averge consumption in society ($\bar{c}$) as a reference point to calculate the relative position. Formally, $u(c, c/(\bar{c}))$ where marginal utility is positive in both arguments and $c/\bar{c}$ is the relative position.

Following this model, we define the relative distribution as dependent not only on the level on consumption but also on the positionality index assigned by each individual to his level of consumption. Therefore $\frac{q_i^\pi(\sigma)}{\sum_{j \in N} q_j^\pi(\sigma)}$ represents the measure of relative concerns that player i assigns to his distribution q_i. Here we present the mathematical expression of the status function as an explicit relation of absolute consumption q, positionality index π and relative consumption as defined above.

$$\delta_i(\sigma, \pi) = \frac{q_i^\pi(\sigma)}{\sum_{j \in N} q_j^\pi(\sigma)} \sum_{j \in N} q_j(\sigma) \tag{1}$$

where $\sigma \in \Sigma$ is point in the joint action set, q_k is player k's absolute consumption, π is the positionality index and N is the total number of players. In relation with the characteristics of the positionality index exposed in the previous section, let us consider the simple example where two individuals have, respectively, the following level of absolute consumptions of some positional good: $(4, 2)$.

With the formulation of status function in 1 it is easy to show that when $\pi = 0$, then relation (2) in [8] emerges: $\delta_1(., 0) = \delta_2(., 0) = 3$, which is exactly the definition of public goods. If $\pi = 1$ then of course the standard private goods consumption levels are obtained, thus $\delta_1(., 1) = 4$ and $\delta_2(., 1) = 2$; and, eventually, if π is sufficiently high $\hat{\pi}$, then a strict competitive distribution is realised, for $\delta_1(., \hat{\pi}) = 6$ and $\delta_2(., \hat{\pi}) = 0$, which is a form of zero-sum game as required by Pagano.

We prove now the theorem that verifies that 1 is equipped with the features we have just defined.

Function 1 satisfies all the properties of a status function.

We shall prove that properties 1-4 hold for 1.

Invariance.

$$\delta_i(\sigma, 1) = \frac{q_i(\sigma)}{\sum_{j \in N} q_j(\sigma)} \sum_{j \in N} u_j(\sigma) = q_i(\sigma)$$

Exclusivity. It is a two-sided relation. Let us begin with the *if* condition:

$$\frac{1}{\sum_{j \in N} q_j(\sigma^0)^\pi} \geq \frac{1}{\sum_{j \in N} q_j(\sigma^1)^\pi}$$

multiplying LHS by $q_i(\sigma^0)^\pi \sum_{j \in N} q_j(\sigma^0)$ and RHS by $q_i(\sigma^1)^\pi \sum_{j \in N} q_j(\sigma^1)$ which are equal we obtain:

$$\delta_i(\sigma^0, \pi) \geq \delta_i(\sigma^1, \pi)$$

And for the *only if* condition:

$$\frac{q_i(\sigma^0)^\pi}{\sum_{j \in N} u_j(\sigma^0)^\pi} \sum_{j \in N} q_j(\sigma^0) \geq \frac{q_i(\sigma^1)^\pi}{\sum_{j \in N} q_j(\sigma^1)^\pi} \sum_{j \in N} q_j(\sigma^1)$$

when $\sum_{j \in N} q_j(\sigma^0) = \sum_{j \in N} q_j(\sigma^1)$ and $q_i(\sigma^0) = q_i(\sigma^1)$, $\forall i \in N$ it reduces to:

$$\sum_{j \in N} q_j(\sigma^0)^\pi \geq \sum_{j \in N} q_j(\sigma^1)^\pi$$

Symmetry.

$$q_i(\sigma) = q_k(\sigma)$$

If we raise both sides to the power of π and multiply by $\frac{\sum_{j\in N} q_j(\sigma)}{\sum_{j\in N} q_j(\sigma)^\pi}$, then we have:

$$\delta_i(\sigma,\pi) = q_i(\sigma)^\pi \frac{\sum_{j\in N} q_j(\sigma)}{\sum_{j\in N} q_j(\sigma)^\pi} = q_k(\sigma)^\pi \frac{\sum_{j\in N} q_j(\sigma)}{\sum_{j\in N} q_j(\sigma)^\pi} = \delta_k(\sigma,\pi)$$

Efficiency

$$\sum_{i\in N} \delta_i(\sigma,\pi) = \sum_{i\in N} \frac{q_i(\sigma)^\pi}{\sum_{j\in N} q_j(\sigma)^\pi} \sum_{i\in N} q_i(\sigma) = 1 \sum_{i\in N} q_i(\sigma)$$

And this concludes the proof.

As aforementioned we prove that the status function defined above is also the unique formulation that possesses the features we required.

Function 1 is the only function that satisfies properties 1-4.

Let us suppose that there exists a function δ' that possesses properties 1-4 and it is different from δ. If this is the case, then by the property of efficiency:

$$\sum_{i\in N} \delta'_{i\in N}(\sigma,\pi) = \sum_{i\in N} \delta_{i\in N}(\sigma,\pi)$$

Let's rule out the case where $\pi = \vec{1}$ (invariance), for in that case $\delta' = \delta$. This leaves us with the following relation:

$$\delta'_l(\sigma,\pi) + \delta'_m(\sigma,\pi) = \delta_l(\sigma,\pi) + \delta_m(\sigma,\pi)$$

Let us suppose that (i) $\delta'_l(\sigma,\pi) > \delta_l(\sigma,\pi)$, and obviously $\delta'_m(\sigma,\pi) < \delta_m(\sigma,\pi)$. Let us also assume that there exist two states $\sigma^1, \sigma^2 \in \Sigma$ such that (ii) $q_j(\sigma^1)^\pi = q_j(\sigma^2)^\pi = q_j(\sigma)^\pi$; therefore we have $\delta'_l(\sigma,\pi) = \delta_l(\sigma^1,\pi) > \delta_l(\sigma,\pi)$ and $\delta'_m(\sigma,\pi) = \delta_m(\sigma^2,\pi) < \delta_m(\sigma,\pi)$. But if this is the case, by (i) and the property of exclusivity we have $\sum q_j(\sigma^2)^\pi < \sum q_j(\sigma)^\pi$ and $\sum q_j(\sigma^1)^\pi > \sum q_j(\sigma)^\pi$, which contradict (ii). This means that all states $\sigma, \sigma^1, \sigma^2$ must have the same exclusivity, therefore $\delta' = \delta$.

So far we have defined a comprehensive tool for the description of positional goods in all their forms (public, private and positional) by means of a simple mathematical relation. Here we show that the formulation in 1 is also featured with the following extra properties:

- **Separation**: Differences in relative distribution of goods with a high positionality index are magnified by the status transformation. (i) if $\exists\sigma \in \Sigma$ such that $q_i(\sigma) - q_k(\sigma) = C > 0$ for some $k \in N$, then $\delta_i(\sigma,\pi) - \delta_k(\sigma,\pi) \geq C$ when $\pi \geq \hat{\pi} \geq 1$. (ii) if $\pi^1 > \pi^0 > \hat{\pi}$ and $\exists\sigma \in \Sigma$ such that $q_i(\sigma) - q_l(\sigma) = C$ for some $k \in N$, then $\delta_i(\sigma,\pi^1) - \delta_i(\sigma,\pi^1) \geq \delta_i(\sigma,\pi^0) - \delta_i(\sigma,\pi^0)$
- **Equality**: Differences in relative distribution of goods with a low positionality index are reduced by the status transformation. (i) if $\exists\sigma \in \Sigma$ such that $q_i(\sigma) - q_k(\sigma) = C > 0$ for some $k \in N$, then $\delta_i(\sigma,\pi) - \delta_k(\sigma,\pi) \leq C$ when $\pi \leq \hat{\pi} \leq 1$. (ii) if $1 > \pi^1 > \pi^0$ and $\exists\sigma \in \Sigma$ such that $q_i(\sigma) - q_l(\sigma) = C$ for some $k \in N$, then $\delta_i(\sigma,\pi^1) - \delta_i(\sigma,\pi^1) \leq \delta_i(\sigma,\pi^0) - \delta_i(\sigma,\pi^0)$

One may easily notice that these extra features are essential to include the complete triad of economic goods within our framework.

Function 1 satisfies the additional properties of *separation* and *equality*

The proof is straightforward.

Separation. Let us write the condition on absolute consumption in the following way: $q_i(\sigma) = q_k(\sigma) - C$. Therefore we can express condition (i) on the status function as:

$$\frac{q_i(\sigma)^\pi - (q_i(\sigma) - C)^\pi}{q_i(\sigma)^\pi - (q_i(\sigma) - C)^\pi}(2q_i(\sigma) - C) \geq C$$
$$q_i(\sigma)^\pi - Cq_i(\sigma)^{\pi-1} \geq (q_i(\sigma) - C)^\pi$$

The latter inequality is verified when (i) $\pi > \hat{\pi}$, for it tends to $2q_i(\sigma) - C \geq C$ which is always true; and (ii) by the properties of asymptotic functions, the LHS of the former inequality increases as π increases.

Equality. We need to invert the verse of the second inequality from the previous property to prove that (i) when $\pi \to 0$ then the LHS of $q_i(\sigma)^\pi - Cq_i(\sigma)^{\pi-1} \leq (q_i(\sigma) - C)^\pi$ tends to $1 - \frac{C}{q_i(\sigma)}$ and the RHS tends to 1 which always verifies the inequality. To prove (ii) we use the same argument we used to prove the second part of *separation*.

Let us consider the example depicted by the game to summarise all the properties of the status function:

$$\begin{pmatrix} 8,8,8 & 7,7,2 \\ 7,5,4 & 12,12,1 \end{pmatrix} \qquad \begin{pmatrix} 6,6,7 & 1,5,12 \\ 12,1,12 & 4,4,4 \end{pmatrix}$$

The properties of invariance and efficiency are straightforward. To show exclusivity let's consider $\sigma^0 = (1,0,1)$ and $\sigma^1 = (0,1,1)$ and a positionality index $\pi > 1$. In these points the condition of exclusivity are met. Additionally, in σ^1 player 1's distribution is much more exclusive than in σ^0 for $\sum_{j\in N} q_j(\sigma^0)^\pi \leq \sum_{j\in N} q_j(\sigma^1)^\pi$. Therefore $\delta_1(\sigma^1,\pi) = \frac{1}{7^\pi+5^\pi+4^\pi} > \frac{1}{7^\pi+7^\pi+2^\pi} = \delta_1(\sigma^0,\pi)$.

We utilise the state $\sigma^1 = (1,1,1)$ to show the property of symmetry since the relative distribution is the same for every player $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and, absolute consumptions are respectively $q_1(\sigma^1) = q_2(\sigma^1) = 8$. Hence $\delta_i(\sigma^1,\pi) = \delta_k(\sigma^1,\pi) = 8$, $\forall i,k,\pi$.

Let us assume we have two positionality indices $\pi_0 = 2$ and $\pi_1 = 5$. In $\sigma^1 = (0,1,1)$ the following condition holds $q_1(\sigma^1) - q_2(\sigma^1) = 7 - 5 = 2$. Then $\delta_1(\sigma^1,\pi_0) - \delta_2(\sigma^1,\pi_0) = \frac{7^2-5^2}{7^2+5^2+4^2}(7+5+4) > 2$. Moreover $\delta_1(\sigma^1,\pi_1) - \delta_2(\sigma^1,\pi_1) = \frac{7^5-5^5}{7^5+5^5+4^5}(7+5+4) > \frac{7^2-5^2}{7^2+5^2+4^2}(7+5+4) > 2$. For the same reason if we invert the inequalities we have in the same point σ^1 the property of equality.

4. Generalised status function

As remarked in the introductory section, it is not always the case that the intensity of externalities produced by an individual's consumption of positional goods is equal to the intensity of the balancing externality perceived by others. Hence the generalisation of the concept of status function 1 to a broader function, which takes as second argument a vector of positionality indeces, one for each player appears to be quite obvious. We define the *generalised status function* as:

$$\delta_i(\sigma,\boldsymbol{\pi}) = \frac{q_i^{\pi_i}(\sigma)}{\sum_{j\in N} q_j^{\pi_j}(\sigma)} \sum_{j\in N} q_j(\sigma) \qquad (2)$$

where $\sigma \in \Sigma$ is point in the joint action set, q_k is player k's absolute consumption, $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ is the vector of individual positionality indeces and N is the total number of participants.

As a conclusive remark on the concept of positionality index, we state that certain values of π identify the type of good and differences in the values of the elements of $\boldsymbol{\pi}$ are measures for the externalities produced by the consumption.

Now that we have setted the formal basis to describe the competition for positional goods, we can start analysing the equilibrium concepts that arise. In the following sections we provide both exact and approximated solutions to the problem presented so far.

5. Equilibrium point

The following theorem guarantees the presence of an equilibrium strategy in the transformed game. We are interested in finding the equilibrium points of the transformed game, because the equilibrium profile of the transformed game represents the stable absolute consumption of a positional good. Hence, we are assured that such profile is also the equilibrium point in absolute consumption. We assumed that $q_i(\sigma)$ is a concave function in σ_i. Therefore the following must hold:

Every game in private goods $G^\delta = \delta(G, \boldsymbol{\pi})$, derived from the transformation of a game in absolute consumption G by means of δ has a Nash equilibrium in pure strategies.

To prove this theorem we must first consider the status function as an application on absolute consumption. Since by construction, the set defined by $q_i(\sigma)$ is compact, for it is the image of a continuous transformation on the compact set Σ, we can apply Glicksberg's fixed point theorem, [14], to assure that the game G^δ has a Nash equilibrium at least in mixed strategy. We are interested in the conditions for which an equilibrium point exists in pure strategies.

δ_i is a non decreasing function in the good allocation $q_i(\sigma)$, for its first derivative with respect to q_i is:

$$\frac{q_i(\sigma)^{\pi_i} + \pi_i q_i(\sigma)^{\pi_i - 1}\left(\sum_{j\in N} q_j(\sigma)\right)}{\sum_{j\in N} q_j(\sigma)^{\pi_j}} - \frac{\pi_i q_i(\sigma)^{2\pi_i - 1}\left(\sum_{j\in N} q_j(\sigma)\right)}{\left(\sum_{j\in N} q_j(\sigma)^{\pi_j}\right)^2} \tag{3}$$

which is always non-negative. Moreover $q_i(\sigma)$ is a concave function. Since every monotone transformation of a concave function is quasiconcave, then $\delta_i(\sigma, \boldsymbol{\pi})$ is quasiconcave in σ_i. If this is the case then a Nash equilibrium in pure strategies always exists.

Given the result in this theorem, the equilibrium point of the non-positional game, σ^*, represents the strategy profile in the positional game $G(\sigma^*)$ in which no player can increase his status by changing the level of absolute consumption when his opponents don't deviate. Our formulation is in accordance with [1] where the status function is required to be concave in the absolute consumption.

The Nash equilibrium of the game represents a stable solution, which means that is self-enforced: no player has interest in deviating from the equilibrium profile. However it might not be efficient. There exists a broad discussion in the literature analysing the non-optimality of the Nash solution.

Perhaps the most well-known non-cooperative game, the Prisoner's Dilemma, has been used in several occasions to exploit the weaknesses of individualistic rationality [15]; another classic example is represented by the Tragedy of the commons in which utility maximisation brings a whole society to collapse.

To reduce these inefficiecies, in the next section we investigate another class of game approximating the procedure depicted in **??**. We state that such an approximation is always possible and we discuss the results of some simulations showing that under certain conditions, a Pareto-optimal solution in absolute consumption can be obtained.

6. Equilibrium approximations

In order to study the approximation of G^δ, we need to define the *finite status function* by considering 1 in the restricted set of joint pure strategies $S \subset \Sigma$, where each point $s = (s_i, s_{-i}) \in S$ is defined using the standard convention. Then we extend this definition, and propose the *expected status function* as:

$$D_i(\sigma, \boldsymbol{\pi}) = \sum_{s_i \in S_i} \sigma(s)\delta_i(s, \boldsymbol{\pi}) \tag{4}$$

where $\sigma(s) = \sigma_i(s_i)\sigma_{-i}(s_{-i}) \in \Sigma$ is the joint probability measure assigned to the pure strategy $s = (s_i, s_{-i}) \in S$. Again $\Sigma = \times_{i\in N}\Sigma_i$ where Σ_i is the $|S_i|$-dimensional simplex. The game represented by D_i is a α-approximation of a game represented by δ_i, where $\|D_i(\sigma, \boldsymbol{\pi}) - \delta_i(\sigma, \boldsymbol{\pi})\| \leq \alpha$. Therefore in this perspective any equilibrium point of G^D is an approximation of the equilibria in G^δ and thus is an approximated solution to the problem of positional competition.

There always exists a α-approximation G^D of a G^δ game.

D_i and δ_i are continuous function defined on a compact set Σ, hence their image is also a compact set which is also a subset of $\mathbb{R}$. If this is the case, then $\alpha \in \mathbb{R}$ is always a positive real number.

All the equilibrium points defined for the α-approximation games are related to the original game in status functions by the Fudenberg-Levine theorem:

If σ^* is a ϵ-equilibrium of the α-approximated game, then it is a $(\epsilon + 2\alpha)$-equilibrium of the original game.

In our case, since 4 allow the calculation of exact solutions to the approximated game (i.e. $\epsilon = 0$), we can conclude that the equilibria of G^D are 2α-equilibria of the original positional game.

With these settings we can apply the classic equilibrium concepts of finite game theory to analyse the optimality conditions of the approximated solution.

6.1. Equilibrium concepts

When $\boldsymbol{\pi}$ is common knowledge then we define a non-cooperative game in which each player maximises his own expected status function, given individuals' positionality index and resource absolute consumptions.

Every α-approximation G^D of a game in the absolute consumption of positional goods has a Nash equilibrium:

$$D_i(\sigma_i^*, \sigma_{-i}^*, \boldsymbol{\pi}) \geq D_i(\sigma_i, \sigma_{-i}^*, \boldsymbol{\pi}) \tag{5}$$

Due to the particular form of the expected status function, the proof is quite straightforward.

$\delta_i(s, \boldsymbol{\pi})$ is a mapping from the set of joint pure strategies S to the set of real numbers. Moreover $D_i(\sigma_i, \sigma_{-i}, \boldsymbol{\pi})$ is concave in $\sigma_i \in \Sigma_i$, hence the conditions of validity of the Nash theorem are satisfied and an equilibrium point always exists.

Here we remark the existence of correlated equilibria not just as a mathematical exercise but because the results of the simulations clearly show a powerful connection between the correlated equilibrium strategy and Pareto-optimality.

Every α-approximation G^D of a game in the absolute consumption of positional goods has a correlated equilibrium:

$$\sum_{\omega\in\Omega} q_i(w)\delta_i(s_i^*, s_{-i}^*, \pi) \geq \sum_{\omega\in\Omega} q_i(w)\delta_i(s_i, s_{-i}^*, \pi) \qquad (6)$$

Every Nash equilibrium is also a correlated equilibrium. If Theorem 6.1 is true then the corollary is also verified.

Let us consider the following example of game in absolute consumption:

$$\begin{pmatrix} 4,2 & 3,3 \\ 1,5 & 1,1 \end{pmatrix}$$

Let us assume for simplicity that $\boldsymbol{\pi} = \pi$ is equal for all players. The game in absolute consumption has only one Nash equilibrium in pure strategies which is $(1,0)$. Let us further assume that both players assign a low positionality index to the positional good at stake; if this is the case, we should expect small differences in their status functions, since the positional competition tends to a competition for public goods. Therefore let us set $\pi = 0$. Thus the matrix of the associated game results:

$$\begin{pmatrix} 3,3 & 3,3 \\ 3,3 & 1,1 \end{pmatrix}$$

This new game has three Nash equilibria in $(1,1)$, $(1,0)$ and $(0,1)$ and, moreover, all equilibria provide the same payoffs to both players. This means exactly that each player in the original game is indifferent to any outcome which gives him a *non-null* allocation.

When we have a generic set of values for $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$, then the competitive dynamics might change. In fact, let us suppose that player 1 assigns a low positionality index the good at stake, $\pi_1 = 1/2$ and player 2 gives a high positional value to the same good, $\pi_2 = 3$. If this is the case, the matrix associated to the game is:

$$6 \times \begin{pmatrix} \frac{4^{1/2}}{4^{1/2}+2^3}, \frac{2^3}{4^{1/2}+2^3} & \frac{3^{1/2}}{3^{1/2}+3^3}, \frac{3^3}{3^{1/2}+3^3} \\ \frac{1}{1+5^3}, \frac{5^3}{1+5^3} & 1/6, 1/6 \end{pmatrix}$$

We can see that there exists only one Nash equilibrium in mixed strategies, which is $(85/100, 35/100)$.

Once again, these results signify that the equilibrium points calculated in G^D are 2α-equilibrium points for the correspondent original games in absolute consumption.

6.1.1. Optimal positionality index

Recalling the studies reported in [6] and, utilising our formulation of the positionality index, we can add a different meaning to π when it is an exogenous parameter provided by an external entity (i.e. society).

In fact, if the positionality index is *a-priori* imposed to the players, then we can model competitive dynamics in the presence of social regulation or practice policies. Depending on the dynamic that a designer desires to implement, the selection of a value of π could result in different choices. Our main focus is in finding the conditions which create an optimal allocation in the absolute consumption.

Other possible dynamics could be: the reduction of inequalities among the relative levels of consumption, the selection of a monopolist or leader coalitions or competition for pure private goods.

In the experimental part of our work we have considered the values of π which maximise the welfarian absolute consumption of goods. Let us take the Prisoner's Dilemma depicted by:

$$\begin{pmatrix} 5,5 & 1,6 \\ 6,1 & 2,2 \end{pmatrix}$$

and consider the situation where $\pi = 0$; with this configuration then the absolute consumption matrix transforms into the following:

$$\begin{pmatrix} 5,5 & 7/2, 7/2 \\ 7/2, 7/2 & 2,2 \end{pmatrix}$$

It is easy to notice that there is only one Nash equilibrium in pure strategies of the game, which is the point $(1,1)$. This point is also Pareto efficient, since there is no other state in which every player can increase his own status, hence $\pi = 0$ is an optimal positionality index.

Here we discuss the results of a series of computer simulations in which the 2α-equilibrium is calculated for a big set of 3×2 games with random levels of absolute consumption. In particular, for each game, we compare the price of stability of the original game in comsumption, where the good is considered non-positional $\pi = 1$, to that of the same game in consumption, where the good is positional and the value of π is statically assigned in order to maximise the welfarian absolute consumption of such goods. The price of stability is used as a measure of efficiency of the solution and in our frameset is defined as:

$$PoS = \frac{\text{value of best equilibrium}}{\text{value of optimal solution}} \geq 1 \qquad (7)$$

Eventually we consider the Nash and the correlated solutions to the generated games.

The simulation setting is as follows:

- 500 3×2 games with rational payoffs comprised between 1 and 20;
- π^* equal for every player from the set $\Pi^* = [0, 20]$ that maximise the total consumption of players, i.e. $\sum_{i\in N} q_i(\sigma^*)$.

Figure 1 compares the values of the PoS, respectively for the Nash and correlated equilibria, in the non-positional game with the PoS of the same game where the optimal positionality index is given.

From these figures it is evident that by conditioning the positionality index it is possible to increment the efficiency of the equilibrium point in absolute consumption, in particular when players have the intention to correlate their strategy, the level of consumption reaches the Pareto optimal frontier. In our view, such a result verifies how public regulations and common policies, which have effect on the value of π, indeed influences the strategic behaviour, as explained in [12], [9] and [11]. We also verified that both equilibrium concepts, i.e. Nash and correlated equilibria, provide a higher value for the total absolute consumption.

6.2. Games with incomplete information

When $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_N)$ is not common knowledge, and has a probability associated to each component, then we define *incomplete information positional game* as a non-cooperative game in which each player maximises his own expected generalised status function given his type and the types of his opponents.

We distinguish two cases within the framework of games with incomplete information: (i) each type is associated to a value of π, hence $\theta_j = \pi_j$ and, (ii), to each type is associated more than one value of π with some probability $p(\pi_k|\theta_j)$.

Extending the notation provided in 2, the expected generalised status function when each type has only one π associated to it, we have: $D_i(s_i, s_{-i}(\theta_{-i}), \theta_i, \theta_{-i})$ where $\theta_j = \pi_j$. This represents the average status function for player i when all other players $-i$ are given a type θ_{-i}. In a similar fashion we define the expected status function where each type is associated with an arbitrary number of π as: $D_i(s_i, s_{-i}(\theta_{-i}, \pi_{-i}(\theta_{-i})), \theta_i, \theta_{-i}, \pi_i(\theta_i), \pi_{-i}(\theta_{-i})$. In this case the formula represents the average status function for player i of type θ_i and with positionality index π_i when all other players $-i$ are given a type θ_{-i} and a positionality index π_{-i} which is associated to the particular type with a known probability.

Hence, we define one Bayesian-Nash equilbrium point in each of the two cases: for case (i):

$$s_i^*(\theta_i) \in \arg\max_{s_i \in S_i} \sum_{\Theta_{-i}} p(\theta_{-i}|\theta_i) D_i(s_i, s_{-i}^*(\theta_{-i}), \theta_i, \theta_{-i}) \quad (8)$$

And for case (ii):

$$s_i^*(\theta_i, \pi_i(\theta_i)) \in \arg\max_{s_i \in S_i} \sum_{\Pi_{-i}} \sum_{\Theta_{-i}} p(\theta_{-i}|\theta_i) p(\pi_{-i}(\theta_{-i})|\theta_{-i}) \\ D_i(s_i, s_{-i}^*(\theta_{-i}, \pi_{-i}(\theta_{-i})), \theta_i, \theta_{-i}, \pi_i(\theta_i), \pi_{-i}(\theta_{-i})) \quad (9)$$

with the natural extension to non-finite types and positionality indeces. In case (i) the existence of a Bayesian-Nash equilibrium point is guaranteed by Harsanyi's Theorem. Additionally it is easy to show that even case (ii) always admits an equilibrium point. Indeed, having a number $|\Theta|$ of types with $|\Pi(\Theta)|$ positionality indeces associated to them, it is equivalent to have a number of types $|\Theta'| = |\Theta| \times |\Pi(\Theta)|$, which is the same settings we have in case (i).

This concludes the discussion on the approximated equilibrium points. We have investigated three different scenarios which required three different equilibrium concepts and, for each of them we showed the consistency of our framework.

In the next section we introduce a novel equilibrium concept, in which each player keeps the information on his own positionality index private. Such a concept is applicable to both approximated and non-approximated game sets. For reasons of clarity we consider examples from the class of approximated games which are more manageable to discuss.

7. Negotiable positionality index

In this section, we consider the situation in which the value of the positionality index is not only private information, but each player i can *opportunistically* select a value of $\pi_i \in \mathbb{R}$ that maximise his own status at the equilibrium. As reported in [9], citing Sen, *the numbers in the payoff matrix can be interpreted simply as welfare indices of the two person and each person's welfare index can incorporate concern for the other*. (italics added).

Indeed, as previously discussed, the theory of utility is broad enough to incorporate other-regarding preferences; however, what we propose in this paper is the explicit relation between absolute consumption of some positional good and the social status that derives from such consumption. It is by following this approach that the equilibrium dynamics discussed here arise.

As an example let us consider the game of Chicken defined by the matrix:

$$\begin{pmatrix} 1,1 & 7,2 \\ 2,7 & 6,6 \end{pmatrix}$$

This game in absolute consumption has two pure strategies Nash equilibria $(1,0)$ and $(0,1)$. To find the non positional corresponding game we should apply the status tranformation with some value of $\boldsymbol{\pi}$. But let us assume that each player i can independently select the value of π_i; let us also assume that no communication on the values of π is allowed. Then, if player 2 chooses $\pi_2 = 0$, then player 1 prefers to play his second row, since:

$$\begin{cases} 3\frac{7^{\pi_1}}{7^{\pi_1}+1} < 4\frac{6^{\pi_1}}{6^{\pi_1}+1} \\ 1 < 9\frac{2^{\pi_1}}{2^{\pi_1}+1} \end{cases} \quad (10)$$

which is true $\forall \pi_1$. However to avoid player 2 to select his first column player 1 should select $\pi_1 = 0$, for if this is the case:

$$\begin{cases} 3\frac{7^{\pi_2}}{7^{\pi_2}+1} < 4\frac{6^{\pi_2}}{6^{\pi_2}+1} \\ 1 < 9\frac{2^{\pi_2}}{2^{\pi_2}+1} \end{cases} \quad (11)$$

which is true $\forall \pi_2$. In conclusion, the point $(\sigma_1^*, \sigma_2^*, \pi_1^*, \pi_2^*) = (0,0,0,0)$ is an equilibrium *combination*. We use the term *combination* to underline the fact that the solution is a combination of strategies and π. Obviously this behaviour becomes visible only if one considers the status tranformation of a game in absolute consumption.

The following theorem guarantees that an equilibrium point in this scenario always exists.

Every game with negotiable positionality index has an equilibrium point:

$$\begin{cases} \delta_i(\sigma_i^*, \sigma_{-i}^*, \boldsymbol{\pi}^*) \geq \delta_i(\sigma_i, \sigma_{-i}^*, \boldsymbol{\pi}^*) \\ \delta_i(\sigma_i^*, \sigma_{-i}^*, \pi_i^*, \pi_{-i}^*) \geq \delta_i(\sigma_i^*, \sigma_{-i}^*, \pi_i, \pi_{-i}^*) \end{cases} \quad (12)$$

This result is true in both approximated and non-approximated games, thus the equilibrium equations 12 would have validity when D_i substitutes δ_i.

If there is no $\boldsymbol{\pi}^* \neq \vec{1}$ that satisfies the second condition of 12, then obviously, for $\boldsymbol{\pi}^* = \vec{1}$, the second condition of 12 is satisfied, hence the problem reduces to finding a Nash equilibrium, which does always exist as shown in Theorem 6.1.

In case each player i's selection of an "appropriate" positionality index is bounded in some closed subset of $\mathbb{R}$, the following corollary holds:

If $\exists \boldsymbol{\pi} \in \bar{B}_r(\vec{0})$, $r \geq 1$ such that conditions in 12 are met, then there exists an equilibrium point.

As an example let us consider the Prisoner's Dilemma defined by the following matrix, with $\bar{B}_r(\vec{0})$ being the 2-dimensional sphere with centre in $(0,0)$ and radius $r = 3$:

$$\begin{pmatrix} 5,5 & 1,6 \\ 6,1 & 2,2 \end{pmatrix}$$

Player 2 would force player 1 to play first row, in order to obtain an outcome of 6 by playing second column. Therefore if player 2 picks $\pi_2 = 0$ then $\forall \pi_1 \in [0,3]$ player 1 prefers to play first row for:

$$\lambda_2 \frac{5^{\pi_1} 10}{5^{\pi_1}+1} + (1-\lambda_2)\frac{1^{\pi_1} 7}{1^{\pi_1}+1} \geq \lambda_2 \frac{6^{\pi_1} 7}{6^{\pi_1}+1} + (1-\lambda_2)\frac{2^{\pi_1} 4}{2^{\pi_1}+1} \tag{13}$$

is verified $\forall \lambda_2 \in [0,1]$. But if player 1 is *forced* to play his first row, he does not want player 2 to play second column, hence he needs to find a value of π_1 that solve the inequality:

$$\lambda_1 \frac{10}{5^{\pi_1}+1} + (1-\lambda_1)\frac{7}{6^{\pi_1}+1} \geq \lambda_1 \frac{7}{1^{\pi_1}+1} + (1-\lambda_1)\frac{4}{2^{\pi_1}+1} \tag{14}$$

For $\pi_1 = 0$, the expression in 14 is verified. The same argument is valid if we start from player 1. At this point no player can increase his expected absolute consumption by changing his positionality index or his strategy, hence the point $(\sigma_1^*, \sigma_2^*, \pi_1^*, \pi_2^*) = (1,1,0,0)$ is an equilibrium combination. If we allow $\bar{B}_r(\vec{0})$ and $r > 5$ then $(\sigma_1^*, \sigma_2^*, \pi_1^*, \pi_2^*)$ is no longer an equilibrium, for player i will not be forced to change his strategy from the classic Nash equilibrium profile.

Eventually, we provide the following lemma whose proof is already contained in the peculiar definition of an equilibrium point with negotiable positionality index.

In a game with negotiable positionality index, there exists an equilibrium point that is at least as efficient as any Nash equilibrium of the game in absolute consumption.

The equilibrium points described by 7 represent a rational response to the boundedness deriving from some constraints on the choice of the positionality index. From [9]: characterising a certain 'object' as a good or a bad, as a private or public good, as a relational or positional good or as a positive or negative externality always critically depends on the subjective value-system of the agents.

As we seen in the previous sections, the positionality index is indeed a measure of the agent's perception of a good he is consuming. Therefore by following Zarri's logics, it should be easy to notice that a full interpretation of the concepts described in this section, is represented by his definition of "extended payoffs", which include preferential moral principles (i.e. the restriction on the selection of π).

This concludes our exposition on the non-cooperative aspects of the positional competition. In the next two sections we describe a cooperative methods for the allocation of positionals good and a coalition-based mechanism that is limited in efficiency by the Nash solution to the bargaining problem.

8. The bargaining solution

As intended by [17], a bargaining situation involves two individuals who have the opportunity to collaborate for mutual benefit; additionally no action taken by one individuals without the consent of the other can affect the well-being of the other one. Again, a solution means a determination of the amount of satisfaction each individual should expect to get from the situation, or, rather, a determination of how much it should be worth to each of these individuals to have this opportunity to bargain.

We define the distribution set $X = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_N) | \sum_{i\in N} x_i = K\}$; obviously X is compact. Let us consider the disagreement point to be $d = \vec{1}$. The status function on the distribution set is then defined as $\delta_i(\boldsymbol{x}, \boldsymbol{\pi}) \in \Delta$. The set Δ is the image of a compact set via a continuous function, hence it is also compact.

If this is the case then we can define the Nash product in [17] for status functions as:

$$f^S(\Delta, d) = \prod_{i\in N} (\delta_i(\boldsymbol{x}, \boldsymbol{\pi}) - 1) \tag{15}$$

To find the bargaining solution to the problem of allocating resources according to the status function is equivalent to solve the following maximisation problem:

$$\max_{\delta_1,\ldots,\delta_N} f^s(\Delta, d) \tag{16}$$
$$\sum_{i\in N} x_i = K$$

We consider only situations in which either every player has a zero positionality index or no player has a zero positionality index, for in all other cases there is no real competition since at least one player has no interests in the consumption of the good. We show that in case where at least one player has a null positionality index, the second order condition for optimality are never met.

A solution to this problem always exists for it is defined on a compact set and f^s is continuous since it is the product of continuous functions. It is interesting to see the dynamics of the bargaining process for different positionality indeces.

Let us analyse the problem with two players: the extension to many players is straightforward. Solutions to this problem should satisfy the first order conditions for optimality, therefore:

$$\frac{\delta_1'(x, \boldsymbol{\pi})}{1 - \delta_1(x, \boldsymbol{\pi})} = \frac{\delta_2'((K-x), \boldsymbol{\pi})}{\delta_2((K-x), \boldsymbol{\pi}) - 1} \tag{17}$$

which reduces to the simple relation: $(K-x)^{\pi_2} = x^{\pi_1}$. And the second order conditions in the stationary points:

$$\delta_1''(x, \boldsymbol{\pi})(\delta_2(x, \boldsymbol{\pi}) - \delta_1(x, \boldsymbol{\pi})) + \delta_1'(x, \boldsymbol{\pi})(\delta_2'(x, \boldsymbol{\pi}) - \delta_1'(x, \boldsymbol{\pi})) < 0 \tag{18}$$

which is equal to $-2\delta_1'(x, \boldsymbol{\pi}) < 0$, additionally, for $\delta_1'(x, \boldsymbol{\pi})$ is positive. The second order condition in case a player has a null positionality index (i.e. $\pi_2 = 0 \neq \pi_1$) reduces to the following relation:

$$\frac{K\pi_1}{x} - \frac{\pi_1}{2} < 0 \tag{19}$$

which resolves to the condition $2K < x$ when $\pi_1 \neq 0$ that is outside of the feasible region.

It is now easy to notice that when $\pi_1 = \pi_2 \neq 0$ then there exists a solution $(K/2, K/2)$ which reflects the idea that if two individuals evaluate the same good in the same manner, they would fairly split the total amount.

In the case where $\pi_1 = \pi_2 = 0$ then every allocation is a solution: again this is a reasonable situation for if both players do not receive positionality by the consumption of the good at stake, then every allocation is optimal.

The last dynamic is when $\pi_1 \neq \pi_2 \neq 0$. In this case, one player gives a higher positionality index to the good, which means that he gets more satisfaction per unit than his opponent: therefore it is not surprising that, in order to meet the condition of optimality, the player with higher positionality index will receive a smaller amount of good.

In the general case with more than two players the solutions satisfy the first order condition: $x_1^{\pi_1} = x_2^{\pi_2} = \ldots = (K - \sum_{j\neq N} x_j)^{\pi_N}$.

9. Coalitions

In [16], proposes a model of competition for status in which status is exogenously distributed and individuals desire to associate with those with high status. If status is yet desirable, [12], then agents would prefer to form coalitions in order to increase their own status.

A coalition forms when some subsets of players cooperatively aggregate their marginal level of consumption.

In particular, coalitions form when some players join their allocations in order to increase the decisional power over a distribution of positional goods. Each player in a coalition owns a quota of goods, which is equal to his marginal level of consumption with respect to the aggregate absolute consumption of the coalition.

According to our definition of generalised status function in 2 we must distinguish two types of coalitions, depending on the value of $\boldsymbol{\pi}$. There exists one-sided and two-sided coalitions.

One-sided coalitions refer to situations with common fixed positionality index described by 1; while two-sided coalitions refers to situations described by 2.

The allocation of good related to the member of a coalition C_L results in:

$$C_L^i = \frac{u_i(\sigma^*)}{\sum_{l\in C_L} u_l(\sigma^*)} \frac{\left(\sum_{l\in C_L} u_l(\sigma^*)\right)^\pi}{\left(\sum_{l\in C_L} u_l(\sigma^*)\right)^\pi + \sum_{m\notin C_L} u_m(\sigma^*)^\pi} \sum_{j\in N} u_j(\sigma^*) \tag{20}$$

While a non-member of C_L would receive an allocation according to the relation:

$$C_{-L}^i = \frac{u_i(\sigma^*)^\pi}{\left(\sum_{l\in C_L} u_l(\sigma^*)\right)^\pi + \sum_{m\notin C_L} u_m(\sigma^*)^\pi} \sum_{j\in N} u_j(\sigma^*) \tag{21}$$

Moreover a coalition is stable if no player can increase his allocation by deviating from the agreement.

As an example of one-sided coalition, let us consider a 3-player situation with fixed positionality index that prescribe the following distribution of the positional good: $[7, 5, 4]$.

If $\pi = 3$, then the coalitions that form are reported in Table 2. C_0 is the null coalition, in the sense that is the solution where there is no agreement. Values in Table 2 refer to the coalition modification of the individuals status functions.

In this example the only two stable coalitions are: C_{23} and C_{123} for no player can increment his status by unilateral deviation from the prescribed allocation.

To show two-sided coalition formation, let us consider a 3-player situation with non fixed positionality index with the same distribution prescribed in the previous example, but with $\boldsymbol{\pi} = (0, 1, 2)$.

The list of two-sided coalition is reported in Table 3:

The notation $C_{i\to j}$ indicates the coalition between player i and j where i accepts to utilise j's positionality index. In this second example the only stable coalition is C_0.

There is a precise relation between the stability of coalitions and the bargaining solution described by 17. In fact the following proposition holds:

If $C_0 = f^s(\Delta, d)$ then C_0 is the only stable coalition.

To prove this theorem it is sufficient to use the property of efficiency of the Nash solution to the bargaining problem. In fact, if $f^s(\Delta, d)$ is efficient, every modification to the current allocation will cause some player to lose status and, eventually, all the resultant coalitions are unstable.

Moreover, since a coalition is a redistribution of a constant allocation of good, it is obviously invariant under the Coase theorem.

10. Discussion and conclusions

In this paper we have proposed a comprehensive framework for the analysis of the competition for positional goods, in the extended meaning of the term. We have shown how to incorporate the information on positionality using the concept of positionality index (π) and, we have characterised the values of such an index which are able to describe the consumption of the three different types of economic goods (private, public and positional). We have provided a relation, the status function δ, for the understanding of the strategic behaviour in the presence of positional goods, and we have analysed the equilibrium strategies in such game forms. Additionally we have defined the approximations of positional games, which are defined by the expected status function D. We have shown some theoretical facts for the α-approximations and then the results of computer simulation that we used to evaluate the impact of the positional competition on the absolute consumption. Eventually we have described a mechanism to increase the efficiency of the distribution of positional goods based on the possibility to create coalitions; again, we have provided upper bounds to the forming of coalitions.

In this conclusive section we underline the property of our framework for which, when $\boldsymbol{\pi} = \vec{1}$, the classic dynamic described in the economic models depicted by the authors in [12] emerge. We believe that our framework is general enough to describe a wide range of strategic behaviours that might deviate from the standard non positional competition, and it has the advantage of being self-contained.

A promising field of application of the models proposed in this work is that of digital economy. The rate at which social interactions are migrating to the digital world is astounding. As mentioned in [18] not only humans are using digital supports to interact with each other, but they are more and more interacting with intelligent algorithms. This multiplies the number of samples that researcher might have access to and produce a consistent base for analytical analysis. If economy goes digital, then our models can serve many purposes, in particular:

- make predictions about human behaviour (that is more measurable as it reduces to human-machine interaction)
- create artificial intelligent algorithms that simulate human perception of economic goods, making them more economically aware subjects.

The future of the digital world suggests that each of us will have an electronic code [18] and, a status function that will allow us to express our preferences and calculate our decisions.

Since the values of π might depend on non-measureable elements, we suggest that defining an effective method to evaluate the positionality index could reduce the risk of loss in predictive power which is described as inevitable by [9] when incorporating relative consumption in economic models. We have given a definition for π which allowed us to show the diverse significance of positionality indices in relation to the externalities produced by consumption, but the description of a practice able to measure positionality is out of the scope of this work.

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