

Minkowski symmetry sets of plane curves

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March 25, 2015

Abstract

We study in this paper the Minkowski Symmetry Set (MSS) of a closed smooth curve γ in the Minkowski plane. We answer the following question which is analogous to one concerning curves in Euclidean plane that was treated in [7]: given a point p on γ , does there exist a bi-tangent pseudo-circle that is tangent to γ at both p and at some other point q on γ ? The answer is yes, but as pseudo-circles with non-zero radii have two branches (connected components) it is possible to refine the above question to the following one: given a point p on γ , does there exist a branch of a pseudo-circle that is tangent to γ at both p and at some other point q on γ ? This question is motivated by the quest in [12] to define the Minkowski Blum medial axis, a counterpart of the Blum medial axis of curves in the Euclidean plane ([3]).

1 Introduction

The symmetry set of a submanifold M in the Euclidean n -space is the locus of centres of hyperspheres tangent to M in at least two distinct points. The symmetry sets (and some of their subsets) of plane curves and of surfaces the Euclidean 3-space are well studied and have applications in computer vision and shape recognition (see for example [13]).

We consider in this paper the Minkowski Symmetry Set (MSS) of a closed smooth curve γ in the Minkowski plane, which is defined as the locus of centres of pseudo-circles

*Supported by the FAPESP post-doctoral grant 2012/05326-9

†Partially supported by the grants FAPESP 2014/00304-2, CNPq 301589/2012-7 and CNPq 472796/2013-5.

2000 Mathematics Subject classification. Primary 53A35; Secondary 53B30 57R45

Key Words and Phrases. Curves, Minkowski plane, symmetry sets, evolutes, caustics, singularities.

bi-tangent to γ in at least two distinct points. There are three types of pseudo-circles in the Minkowski plane and points on γ can be spacelike, timelike or lightlike (see §2). Also, the pseudo-circles with non-zero radii have two branches. All of these possibilities make the *MSS* richer than its Euclidean counterpart.

For closed curves in the Euclidean plane, there is a subset of the symmetry set which is of particular interest called the Blum medial axis ([3]). This is defined to be the locus of centres of bi-tangent circles which are completely contained in, γ . The Blum medial axis has the property that it can be used to reconstruct the curve γ (see for example [13]). A Minkowski analogue of the Blum medial axis is defined in [12]. The Minkowski Blum medial axis is the locus of centres of bi-tangent pseudo-circles to γ with the points of tangency lying in just one branch of the pseudo-circles. We call these pseudo-circles 1-branch bi-tangent pseudo-circles.

We answer positively in Theorem 4.1 the following question: given a point p on γ , does there exist a bi-tangent pseudo-circle that is tangent to γ at p and at some other point q ? We also consider the same question but restrict to 1-branch bi-tangent pseudo-circles. We show in Theorem 4.8 that, under some restrictions, given a point p on a spacelike or timelike component C of γ , there is a 1-branch bi-tangent pseudo-circle tangent to γ at p and at another point $q \in C$.

Some brief preliminaries are given in §2. In §3 the properties of the *MSS* and of the caustic at finitely determined singularities of the distance-squared function are studied. The results in §3 are used in §4 for answering the existence of bi-tangent pseudo-circles questions.

2 Preliminaries

The *Minkowski plane* $(\mathbb{R}_1^2, \langle, \rangle)$ is the vector space \mathbb{R}^2 endowed with the pseudo-scalar product $\langle \mathbf{u}, \mathbf{v} \rangle = -u_0v_0 + u_1v_1$, for any $\mathbf{u} = (u_0, u_1)$ and $\mathbf{v} = (v_0, v_1)$. A vector $\mathbf{u} \in \mathbb{R}_1^2$ is called

spacelike if $\langle \mathbf{u}, \mathbf{u} \rangle > 0$,

timelike if $\langle \mathbf{u}, \mathbf{u} \rangle < 0$,

lightlike if $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

The norm of \mathbf{u} is defined by $\|\mathbf{u}\| = \sqrt{|\langle \mathbf{u}, \mathbf{u} \rangle|}$. In all the paper, we refer to a pseudo-unit vector \mathbf{u} , with $\|\mathbf{u}\| = 1$, as a unit vector.

The pseudo-circles in \mathbb{R}_1^2 with centre $c \in \mathbb{R}_1^2$ and radius $r > 0$ are defined as follows:

$$\begin{aligned} H^1(c, -r) &= \{p \in \mathbb{R}_1^2 \mid \langle p - c, p - c \rangle = -r^2\}, \\ S_1^1(c, r) &= \{p \in \mathbb{R}_1^2 \mid \langle p - c, p - c \rangle = r^2\}, \\ LC^*(c) &= \{p \in \mathbb{R}_1^2 \setminus 0 \mid \langle p - c, p - c \rangle = 0\}. \end{aligned}$$

Observe that $LC^*(c)$ is the union of the two lines through c with tangent directions $(1, 1)$ and $(1, -1)$, with the point c removed. The pseudo-circle $H^1(c, -r)$

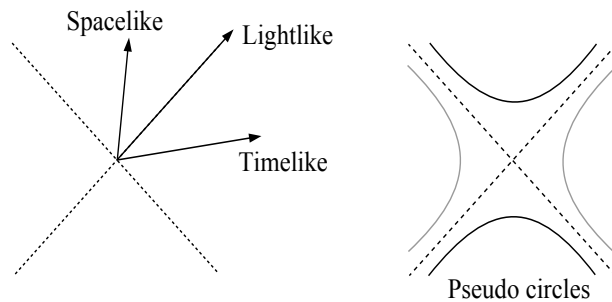


Figure 1: The three types of vectors (left) and pseudo-circles (right) in \mathbb{R}_1^2 .

has two branches which can be parametrised by $c + (\pm r \cosh(t), r \sinh(t))$, $t \in \mathbb{R}$. The pseudo-circle $S^1(c, r)$ also has two branches which can be parametrised by $c + (r \sinh(t), \pm r \cosh(t))$, $t \in \mathbb{R}$. See Figure 1.

Let $\gamma : S^1 \rightarrow \mathbb{R}_1^2$ be a smooth (C^∞) immersion, where S^1 is the Euclidean unit circle. We call the curve γ the image of the map γ and say that it is a closed smooth curve.

The curve γ at t_0 is said to be spacelike if $\gamma'(t_0)$ is spacelike and it is said to be timelike if $\gamma'(t_0)$ timelike. These are open properties so there is a neighbourhood of t_0 where the curve is either spacelike or timelike. If $\gamma'(t_0)$ is lightlike then $\gamma(t_0)$ is said to be a lightlike point. It is shown in ([14], Proposition 2.1) that the set of lightlike points of γ is the union of at least four disjoint non-empty and closed subsets of γ (Figure 2). The complement of these sets are disjoint connected spacelike or timelike pieces of the curve γ .

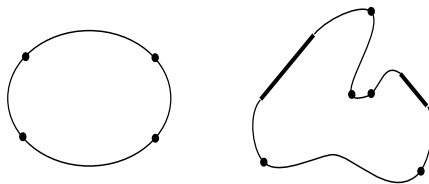


Figure 2: Lightlike points as dots or thick lines on a smooth closed curve in \mathbb{R}_1^2 .

We call the restriction of γ to an open interval (λ, μ) of S^1 , where λ and μ correspond to lightlike points of γ , a *spacelike component* if γ is spacelike in (λ, μ) and a *timelike component* if γ is timelike in (λ, μ) . Thus (λ, μ) is a maximal interval where γ is spacelike or timelike.

The spacelike and timelike components of γ can be parametrised by arc-length. Suppose that $\gamma(s)$, $s \in (\lambda, \mu)$, is an arc-length parametrisation of a component of

γ . Then $\mathbf{t}(s) = \gamma'(s)$ is a unit tangent vector and $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$, where $\kappa(s)$ is the Minkowski curvature of γ at s and \mathbf{n} is the unit Minkowski normal vector at s . The tangent and unit Minkowski normal vectors are pseudo-orthogonal so they are of different types, that is, one is spacelike and the other is timelike or vice-versa.

The evolute of a spacelike or timelike component of γ is the image of the map

$$e(s) = \gamma(s) - \frac{1}{\kappa(s)}\mathbf{n}(s).$$

Observe that the evolute goes to infinity at inflection points (i.e. points where the curvature κ vanishes), so we shall exclude such points when analysing the evolute of a spacelike or timelike component of γ .

In general, the curvature tends to infinity as s tends to λ or μ and the evolute of the curve γ is not defined at the lightlike points. However, the caustic of γ is defined everywhere and contains the evolute of γ ([14]). The caustic can be defined via the *the family of distance-squared functions* $f : S^1 \times \mathbb{R}_1^2 \rightarrow \mathbb{R}$ on γ given by

$$f(t, c) = \langle \gamma(t) - c, \gamma(t) - c \rangle.$$

Denote by $f_c : S^1 \rightarrow \mathbb{R}$ the function given by $f_c(t) = f(t, c)$. We say that f_c has an A_k -singularity at t_0 if $f'_c(t_0) = f''_c(t_0) = \dots = f_c^{(k)}(t_0) = 0$ and $f_c^{(k+1)}(t_0) \neq 0$. This is equivalent to the existence of a local re-parametrisation h of γ at t_0 such that $(f_c \circ h)(t) = \pm(t - t_0)^{k+1}$.

Geometrically, f_c has an A_k -singularity if and only if the curve γ has contact of order $k + 1$ at $\gamma(t_0)$ with the pseudo-circle of centre c and radius $r = \|\gamma(t_0) - c\|$. Denote this pseudo-circle by $C(c, r)$. Thus, the curve γ has order of contact 1 with a pseudo-circle at t_0 if it intersects transversally the pseudo-circle at $\gamma(t_0)$. The order of contact is 2 if the circle and the curve have ordinary tangency at $\gamma(t_0)$.

The caustic of γ is the local component \mathcal{B}_1 of the bifurcation set of the family f , given by

$$\mathcal{B}_1 = \{c \in \mathbb{R}_1^2 \mid \exists t \in S^1 \text{ such that } f'_c(t) = f''_c(t) = 0\}.$$

This is the set of points $c \in \mathbb{R}_1^2$ such that the germ f_c has a degenerate singularity at some point t . The caustic of γ is defined at all points on γ including its lightlike points. The caustic of a generic curve γ is a smooth curve at the lightlike points of γ and has ordinary tangency with γ at such points, see [14] and Figure 3.

3 The Minkowski Symmetry Set

The multi-local component of the bifurcation set of the family f is defined as

$$\mathcal{B}_2 = \{c \in \mathbb{R}_1^2 \mid \exists t_1, t_2 \text{ such that } t_1 \neq t_2, f_c(t_1) = f_c(t_2), f'_c(t_1) = f'_c(t_2) = 0\}.$$

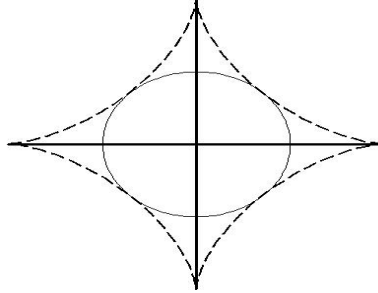


Figure 3: The caustic of an ellipse (dashed) in \mathbb{R}_1^2 : the points of tangency of the caustic with the ellipse are the lightlike points of the ellipse. The thick lines are the Minkowski Symmetry Set of the ellipse.

The full-bifurcation set of f is defined as

$$\text{Bif}(f) = \mathcal{B}_1 \cup \mathcal{B}_2.$$

Definition 3.1 *The Minkowski Symmetry Set of γ (MSS for short) is the locus of centres of pseudo-circles which are*

- (1) *bi-tangent to γ in at least two distinct points p and q , then the pair p, q is called a bi-tangent pair, or*
- (2) *tangent to γ at a single point with contact of order at least 4.*

It is clear that the *MSS* is precisely the closure of the multi-local component \mathcal{B}_2 of the bifurcation set of the family of distance-squared function f on γ . (In Definition 3.1, we allow limit points of the *MSS* where the bi-tangent points coincide to belong to the *MSS*.)

It follows from Thom's transversality theorem (see for example [4, 8]) that for an open and dense set of immersions $\gamma : S^1 \rightarrow \mathbb{R}_1^2$ the function f_c has only local singularities of type A_1, A_2, A_3 and multi-local singularities of type A_1^2, A_1A_2, A_1^3 . The geometric characterisation of the local singularity of f_c at a non-lightlike point is as in Table 1. At a lightlike point, the function f_c has generically an A_1 -singularity ([14]) and the curve and the caustic are on different sides of their common tangent line (see Figure 3).

An application of Thom's transversality theorem (see for example Theorem 1 in [5]) also asserts that, for generic curves, the family f is a versal unfolding of the generic singularities of f_c (see for example the appendix in [5] for more details on versal unfoldings). In fact, we have the following result (the proof is technical and is included for completeness. The importance of the result is given in Corollary 3.3).

Table 1: The geometric characterisation of the local singularity of f_c at a non-lightlike point.

The singularity of f_c	The geometric conditions
A_0	γ transversally intersects $C(c, r)$ at $\gamma(t_0)$
A_1	c is on the normal line to γ at $\gamma(t_0)$ but not on its evolute
A_2	c is on the evolute of γ and $\kappa'(t_0) \neq 0$
A_3	c is on the evolute of γ , $\kappa'(t_0) = 0$, $\kappa''(t_0) \neq 0$

Theorem 3.2 *The family f is always a versal unfolding of the generic singularities $A_1, A_2, A_3, A_1^2, A_1A_2$ and A_1^3 of f_{c_0} .*

Proof The A_1 -singularity is stable so is automatically versally unfolded by f . For the local A_k -singularity, $k = 2, 3$, of f_{c_0} which we shall assume to be at $t = 0$, we use the following criterion for showing that f is a versal unfolding of f_{c_0} (see 6.10p in [4]). Write $c_0 = (a_0, b_0)$, $\gamma(t) = (x(t), y(t))$ and

$$\begin{aligned} j^{k-1} \frac{\partial f}{\partial a} (0, c_0) &= \alpha_{1,1}t + \alpha_{1,2}t^2 + \dots + \alpha_{1,k-1}t^{k-1}, \\ j^{k-1} \frac{\partial f}{\partial b} (0, c_0) &= \alpha_{2,1}t + \alpha_{2,2}t^2 + \dots + \alpha_{2,k-1}t^{k-1}, \end{aligned}$$

where $j^l g$ denotes the Taylor polynomial of g of degree l , at $t = 0$, without the constant term. Then f is a versal unfolding of the A_k -singularity of f_{c_0} if and only if the matrix

$$J_{A_k} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,k-1} \\ \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,k-1} \end{pmatrix}$$

has rank $k - 1$. We have

$$J_{A_2} = \begin{pmatrix} -2x'(0) \\ 2y'(0) \end{pmatrix}$$

which has rank 1 as the curve γ is regular and

$$J_{A_3} = \begin{pmatrix} -2x'(0) & -x''(0) \\ 2y'(0) & y''(0) \end{pmatrix}$$

which has rank 2 as the curvature of γ is assumed to be non-zero. (As observed above, the function f_{c_0} has generically only an A_1 -singularity at a lightlike point, so the A_3 -singularities occur generically on a timelike or a spacelike component of γ .)

For the multi-local singularities, we use the criterion in the appendix in [5]. For the A_1^2 -singularity, say at t_1 and t_2 , the family f is a versal unfolding of f_{c_0} if and only if

$$\begin{pmatrix} x(t_2) - x(t_1) \\ y(t_2) - y(t_1) \end{pmatrix}$$

has maximal rank, which it has as the two points $\gamma(t_1)$ and $\gamma(t_2)$ are distinct. For the A_1A_2 -singularity, we require

$$\begin{pmatrix} x(t_2) - x(t_1) & x'(t_2) \\ y(t_2) - y(t_1) & y'(t_2) \end{pmatrix}$$

to have maximal rank. This happens if and only if the tangent direction to γ at the A_2 -singularity t_2 is not parallel to $\gamma(t_1) - \gamma(t_2)$, alternatively, when the tangent line to γ at t_2 does not pass through $\gamma(t_1)$. But this is always true as $\gamma(t_1)$ and $\gamma(t_2)$ also belong to a pseudo-circle. Therefore f is always a versal unfolding of the A_1A_2 -singularity of f_{c_0} .

For the A_1^3 -singularity, say at t_1, t_2 and t_3 , f is a versal unfolding if and only if

$$\begin{pmatrix} x(t_2) - x(t_1) & x(t_3) - x(t_1) \\ y(t_2) - y(t_1) & x(t_3) - x(t_1) \end{pmatrix}$$

has maximal rank. This is always the case as the three points $\gamma(t_1), \gamma(t_2)$ and $\gamma(t_3)$ are also on a pseudo-circle so cannot be collinear. \square

Corollary 3.3 *The evolute and the MSS of a generic curve γ in the Minkowski plane are diffeomorphic to one of the models in Figure 4.*

Proof The proof follows from the fact that two versal families of the same singularity type and with the same number of parameters have diffeomorphic bifurcation sets. The bifurcation sets in Figure 4 are those of the model versal families of the singularities in Theorem 3.2. \square

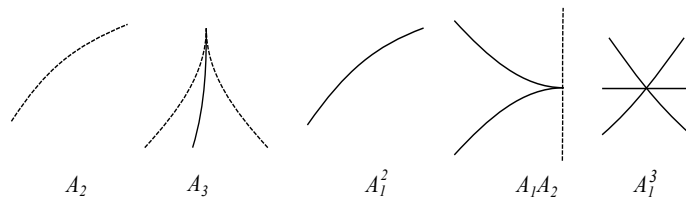


Figure 4: Generic models of the evolute (dashed) and of the MSS (continuous).

The function f_c , with c the intersection of two lightlike tangent lines of γ at $\gamma(t_1)$ and $\gamma(t_2)$, has generically an A_1^2 -singularity at these points. The MSS is a smooth curve at c and has tangent direction along $\gamma(t_1) + \gamma(t_2)$ at that point ([14]).

We consider now the structure of the caustic and of the MSS at a non-generic singularity of f_c . We start with the case when the singular point is not lightlike, so the caustic coincides with the evolute.

Theorem 3.4 *Let $p = \gamma(t_0)$ be a non-lightlike point and suppose that f_c has an A_k -singularity at t_0 , with $k \geq 3$.*

(i) *The evolute can be parametrised locally in a suitable system of coordinates in the form $(t^{k-1}, t^k \eta(t))$, with $\eta(0) \neq 0$.*

(ii) *The MSS is locally empty if k is even and consists of a smooth branch ending at the singular point of the evolute if k is odd. When k is odd, the branch of the MSS can be parametrised in a suitable system of coordinates in the form $(t^{k-1}, t^{k+1} \xi(t))$, with $\xi(0) \neq 0$.*

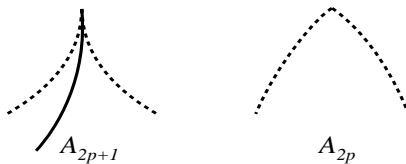


Figure 5: Models of the evolute (dashed) and MSS (continuous) at higher vertices.

Proof (i) Take γ parametrised by arc-length so that its evolute can be parametrised by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)} \mathbf{n}(t)$$

where $\mathbf{n}(t)$ is the unit normal vector such that $(\mathbf{t}(t), \mathbf{n}(t))$ form a positive basis. One can show by induction that the function $f_{e(t_0)}$ has an A_k -singularity at t_0 , with $k \geq 3$, if and only if $\kappa'(t_0) = \kappa''(t_0) = \dots = \kappa^{(k-2)}(t_0) = 0$ and $\kappa^{(k-1)}(t_0) \neq 0$ (we call such a point is a $(k-2)$ -vertex). Then the result follows from the fact that $e(t)$ can be written in the coordinate system with origin the $(k-2)$ -vertex $e(t_0)$ and basis $(\mathbf{t}(t_0), \mathbf{n}(t_0))$, in the form

$$\left(\frac{\kappa^{(k-1)}(t_0)}{\kappa(t_0)} (t - t_0)^k + \dots, \frac{\kappa^{(k-1)}(t_0)}{\kappa(t_0)^2} (t - t_0)^{k-1} + \dots \right).$$

In particular, the singularity of the defining equation of the evolute is of type A_2 (i.e. it is equivalent by smooth changes of coordinates in the source to $x^2 + y^3 = 0$) if $k = 3$, of type E_6 (i.e. equivalent to $x^3 + y^4 = 0$) if $k = 4$, and has a non-simple singularity if $k > 4$ (see [1] for terminology).

(ii) Suppose that γ is timelike (the spacelike case follows in a similar way). We make a Lorentz transformation in the target and a reparametrisation in the source so that γ is written locally at $t_0 = 0$ in the form $\gamma(t) = (t, \beta(t))$, with

$$\beta(t) = \beta_2 t^2 + \beta_3 t^3 + \dots + \beta_k t^k + h.o.t.$$

Now $\beta_2 \neq 0$ as $\kappa(t_0) \neq 0$. The centre of the pseudo-circle with degenerate contact with γ at the origin is $(0, -1/(2\beta_2))$. We take the nearby centres in the form $c = (a, b - 1/(2\beta_2))$, with a, b near zero and consider the germ of the family of distance-squared functions $f : \mathbb{R} \times \mathbb{R}^2, (0, 0) \rightarrow \mathbb{R}$, given by

$$\begin{aligned} f(t, (a, b)) &= -(t - a)^2 + (\beta(t) - (b - \frac{1}{2\beta_2}))^2 \\ &= -a^2 + (b - \frac{1}{2\beta_2})^2 + 2at - 2b\beta(t) + \phi(t) \end{aligned}$$

with $\phi(t) = -t^2 + \beta(t)^2 + \beta(t)/\beta_2$. We have $\phi(t) = f_0(t) - 1/(4\beta_2^2)$, so

$$\phi(t) = \phi_{k+1}t^{k+1} + h.o.t.,$$

with $\phi_{k+1} \neq 0$ as f_0 is assumed to have an A_k -singularity at $t = 0$.

We analyse the existence of bi-tangent pairs (t_1, t_2) near $(0, 0)$. The pair (t_1, t_2) is a bi-tangent pair if there exist (a, b) such that

$$\begin{aligned} \frac{\partial f}{\partial t}(t_1, (a, b)) &= 0, \\ \frac{\partial f}{\partial t}(t_2, (a, b)) &= 0, \\ f(t_1, (a, b)) &= f(t_2, (a, b)). \end{aligned} \tag{1}$$

Equivalently,

$$\begin{aligned} a - b\beta'(t_1) + \frac{1}{2}\phi'(t_1) &= 0, \\ a - b\beta'(t_2) + \frac{1}{2}\phi'(t_2) &= 0, \\ a - b\frac{\beta(t_1) - \beta(t_2)}{t_1 - t_2} + \frac{\phi(t_1) - \phi(t_2)}{2(t_1 - t_2)} &= 0. \end{aligned}$$

Solving the first two equations in a and b gives

$$\begin{aligned} a &= \frac{\beta'(t_2)\phi'(t_1) - \beta'(t_1)\phi'(t_2)}{2(\beta'(t_1) - \beta'(t_2))}, \\ b &= \frac{\phi'(t_1) - \phi'(t_2)}{2(\beta'(t_1) - \beta'(t_2))}. \end{aligned}$$

Substituting into the third equation gives an expression of the form

$$\frac{g(t_1, t_2)}{2(t_1 - t_2)(\beta'(t_1) - \beta'(t_2))} = 0, \tag{2}$$

with

$$\begin{aligned} g(t_1, t_2) &= (t_1 - t_2)(\beta'(t_2)\phi'(t_1) - \beta'(t_1)\phi'(t_2)) - (\beta(t_1) - \beta(t_2))(\phi'(t_1) - \phi'(t_2)) \\ &\quad + (\beta'(t_1) - \beta'(t_2))(\phi(t_1) - \phi(t_2)). \end{aligned}$$

It can be shown, by computing the successive partial derivatives of g , that

$$g(t_1, t_1) = \frac{\partial g}{\partial t_1}(t_1, t_1) = \frac{\partial^2 g}{\partial t_1^2}(t_1, t_1) = \frac{\partial^3 g}{\partial t_1^3}(t_1, t_1) = 0$$

and

$$\frac{\partial^4 g}{\partial t_1^4}(t_1, t_1) = 4k(k^2 - 1)(k - 2)\beta_2\phi_{k+1}t_1^{k-2} + h.o.t.$$

Therefore, we can write $g(t_1, t_2) = (t_1 - t_2)^4 \tilde{g}(t_1, t_2)$ for some germ of a smooth function \tilde{g} which does not vanish at $t_1 = t_2$ when $t_1 \neq 0$. Similarly, the denominator in (2) can be written in the form $(t_1 - t_2)^2 \tilde{\beta}(t_1, t_2)$, with $\tilde{\beta}$ non-vanishing at $t_1 = t_2$ when $t_1 \neq 0$ as $\beta_2 \neq 0$.

The points on the diagonal $t_1 = t_2$ do not give bi-tangent pairs, so the bi-tangent pairs are given by the zero set of \tilde{g} . We determine now the structure of the zero set of \tilde{g} .

Write the $(k + 2)$ -jet of g in the form

$$j^{k+2}g(t_1, t_2) = \beta_2\phi_{k+1}(t_1 - t_2)P_k(t_1, t_2),$$

with P_k the homogeneous polynomial

$$P_k(t_1, t_2) = (1 - k)t_1^{k+1} + (1 + k)t_1^k t_2 - (1 + k)t_1 t_2^k - (1 - k)t_2^{k+1}.$$

We claim that $t_1 = t_2$ is the only repeated factor of P_k . Indeed, suppose that $t_1 = \alpha t_2$ is a repeated factor of P_k . Then $P_k(\alpha t_2, t_2) = \partial P_k / \partial t_1(\alpha t_2, t_2) = 0$, that is

$$(1 - k)\alpha^{k+1} + (1 + k)\alpha^k - (1 + k)\alpha - (1 - k) = 0, \quad (3)$$

$$(1 - k)\alpha^k + k\alpha^{k-1} - 1 = 0. \quad (4)$$

The linear combination $[-(k + (1 - k)\alpha)](3) + [(1 - k)\alpha^2 + (1 + k)\alpha](4)$ of equations (3) and (4) gives

$$k(1 - k)(\alpha - 1)^2 = 0,$$

so $\alpha = 1$.

Therefore, $j^{k-2}\tilde{g}$ is a homogeneous polynomial with no repeated (complex or real) factors. It follows that the ideal generated by the partial derivatives of \tilde{g} is Newton non-degenerate (see [11, 2] for terminology) so has finite codimension. This implies that there exists a germ of a diffeomorphism in the source such that $\tilde{g} \circ h = j^l \tilde{g}$ for some $l \geq k - 2$. Thus, the zero set of \tilde{g} is diffeomorphic to that of $j^l \tilde{g}$ (which is an algebraic curve). The number of branches, at the origin, of $j^l \tilde{g} = 0$ are determined by its tangent cone, i.e., by the number of solutions of $j^{k-2}\tilde{g} = 0$, and as these are all simple, the branches are all smooth curves. We now determine the number of these curves.

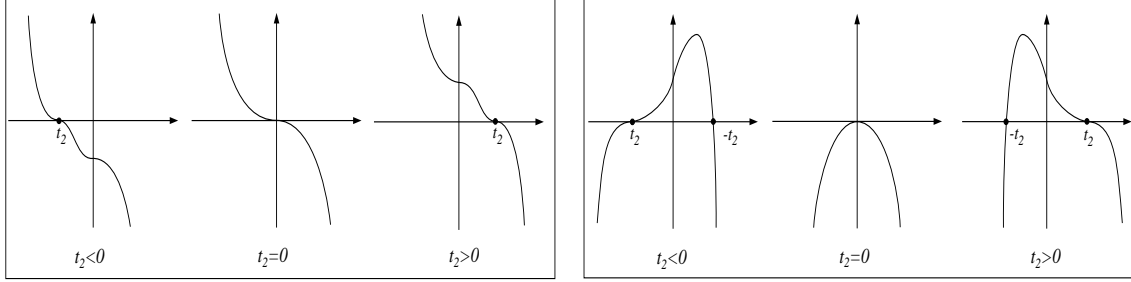


Figure 6: Graphs of the function $f(t_1) = P_k(t_1, t_2)$ for t_2 fixed, left when k is even and right when k is odd.

It is clear that $t_1 + t_2$ is a factor of P_k if and only if k is odd, so using the fact that $(t_1 - t_2)^4$ is a factor of g , we have

$$P_k(t_1, t_2) = \begin{cases} (t_1 - t_2)^3(t_1 + t_2)J_k(t_1, t_2) & \text{if } k \text{ is odd} \\ (t_1 - t_2)^3L_k(t_1, t_2) & \text{if } k \text{ is even} \end{cases}$$

where $J_k(t_1, t_2)$ and $L_k(t_1, t_2)$ are homogeneous polynomials with no repeated factors.

To show that L_k and J_k have no real factors, we consider the function $f(t_1) = P_k(t_1, t_2)$ with t_2 fixed and draw its graph. Elementary calculations show that these graphs are as in Figure 6, where the inflections at $t_1 = t_2$ are ordinary ones. Therefore, L_k and J_k vanish only at $(0, 0)$, i.e., they have no real factors. Therefore, there exists one smooth branch of $\tilde{g} = 0$ if k is odd and none if k is even. When k is odd, the solution of $\tilde{g} = 0$ has initial term $t_2 = -t_1 + h.o.t$, so the *MSS* has a parametrisation of the form

$$(a, b) = \left(-\frac{(k+2)\phi_{k+2}}{2}t_1^{k+1} + h.o.t, \frac{(k+1)\phi_{k+1}}{4\beta_2}t_1^{k-1} + h.o.t \right).$$

□

Remark 3.5 When $k = 2l$, the polynomial L_k in the proof of Theorem 3.4 is in fact the following self-reciprocal polynomial of degree $2l - 2$

$$L_{2l} = -(a_0t_1^{2l-2} + \dots + a_{l-2}t_1^l t_2^{l-2} + a_{l-1}t_1^{l-1} t_2^{l-1} + a_{l-2}t_1^{l-2} t_2^{l+1} + \dots + a_0t_2^{2l-2})$$

with

$$a_i = l^2 - (l - 1 - i)^2, \quad i = 0, \dots, l - 1.$$

When $k = 2l + 1$, the polynomial J_{2l+1} in the proof of Theorem 3.4 is also a self-reciprocal polynomial of degree $2l - 2$, given by

$$J_{2l+1} = b_0t_1^{2l-2} + \dots + b_{l-2}t_1^l t_2^{l-2} + b_{l-1}t_1^{l-1} t_2^{l-1} + b_{l-2}t_1^{l-2} t_2^{l+1} + \dots + b_0t_2^{2l-2}$$

with

$$b_0 = l, b_1 = l - 1, b_2 = 2(l - 1), b_3 = 2(l - 2), b_i = 2b_{i-1} - 2b_{i-2} + b_{i-3}, i = 4, \dots, l - 1.$$

The proof of Theorem 3.4 shows that L_{2l} and J_{2l+1} have no real roots (see for example [10] on the zeros of self-reciprocal polynomials). In fact, the polynomials L_{2l} and J_{2l+1} are self-reciprocal because $j^{k+2}g$ is a self-reciprocal polynomial. Indeed, if (t_1, t_2) is a bi-tangent pair so is (t_2, t_1) .

Consider now the case when the distance-squared function has a degenerate singularity at a lightlike point.

Theorem 3.6 *Suppose that $p = \gamma(t_0)$ is a lightlike point and that f_{c_0} has an A_k -singularity at p .*

- (i) *If $k \geq 2$, the caustic of γ is the union of the lightlike line tangent to γ at p and a smooth curve which has $(k + 1)$ -order of contact with γ at p .*
- (ii) *The MSS is empty if k is odd and it consists of two smooth curves tangent to γ at p if k is even.*

Proof (i) Take p to be the origin and the tangent line to γ at p to be parallel to the direction $(1, 1)$; the case where the tangent line is parallel to $(1, -1)$ follows similarly. The curve can then be parametrised locally in the form $\gamma(t) = (t, t + \beta(t))$, where β is a germ at 0 of a smooth function with $\beta(0) = \beta'(0) = 0$. We write $c = (a, b)$ so that $f(t, c) = -(t - a)^2 + (t + \beta(t) - b)^2$. By hypothesis, the origin is an A_k -singularity of some distance-squared function f_{c_0} with $c_0 = (a_0, b_0)$ and $b_0 = a_0$. Thus, $\beta(t) = \beta_{k+1}t^{k+1} + h.o.t$, with $\beta_{k+1} \neq 0$.

The centre c belongs to the caustic of γ if and only if there exists t such that $\frac{\partial f}{\partial t}(t, c) = \frac{\partial^2 f}{\partial t^2}(t, c) = 0$. Solving this system for a and b gives, for $t \neq 0$,

$$\begin{aligned} b &= b(t) = (2\beta'(t) + t\beta''(t) + \beta''(t)\beta(t) + \beta'(t)^2)/\beta''(t) = (2/k)t + h(t) \\ a &= a(t) = b(t) + b(t)\beta'(t) - \beta(t) - t\beta'(t) - \beta'(t)\beta(t) \end{aligned}$$

for some germ of a smooth function h with $h(0) = h'(0) = 0$.

Therefore, the caustic contains the germ, at the lightlike point p , of the smooth curve C parametrised by $c(t) = (a(t), b(t))$. If $\beta''(0) \neq 0$, the smooth curve C is the entire caustic. When $\beta''(0) = 0$, the lightlike line $a - b = 0$ is a solution of the system of equations giving the caustic for $t = 0$, so is also part of the caustic. (The determination of the order of contact between γ and its caustic follows by standard calculations.)

- (ii) We proceed as in the proof of Theorem 3.4. The pair (t_1, t_2) , with $t_1 \neq t_2$, is a bi-tangent pair if there exists $c = (a, b)$ for which the system of three equations (1)

is satisfied, where f is as in the case (i) above. We make a change of variables in the parameters plane and set $u = a - b$ and $v = b$, so system (1) becomes

$$u - v\beta'(t_1) + \beta(t_1) + t_1\beta'(t_1) + \beta'(t_1)\beta(t_1) = 0 \quad (5)$$

$$u - v\beta'(t_2) + \beta(t_2) + t_2\beta'(t_2) + \beta'(t_2)\beta(t_2) = 0 \quad (6)$$

$$u(t_2 - t_1) - v(\beta(t_2) - \beta(t_1)) + t_2\beta(t_2) - t_1\beta(t_1) + \beta^2(t_2) - \beta^2(t_1) = 0. \quad (7)$$

Equations (5) and (6) give

$$v = v(t_1, t_2) = \frac{\beta(t_2) - \beta(t_1) + t_2\beta'(t_2) - t_1\beta'(t_1) + \beta'(t_2)\beta(t_2) - \beta'(t_1)\beta(t_1)}{\beta'(t_2) - \beta'(t_1)}. \quad (8)$$

Denote the numerator of $v(t_1, t_2)$ in (8) by $\xi(t_1, t_2)$ and its denominator by $\eta(t_1, t_2)$. Substituting v by $v(t_1, t_2)$ in (5), gives

$$u = u(t_1, t_2) = \frac{\xi(t_1, t_2)}{\eta(t_1, t_2)}\beta'(t_1) - \beta(t_1) - t_1\beta'(t_1) - \beta'(t_1)\beta(t_1). \quad (9)$$

Substituting v by $v(t_1, t_2)$ and u by $u(t_1, t_2)$ in (7) and eliminating the denominator gives an equation of the form $g(t_1, t_2) = 0$. The rest of the proof follows in a similar way to that of Theorem 3.4. We show that $g(t_1, t_2) = (t_1 - t_2)^4 \tilde{g}(t_1, t_2)$ for some germ of a smooth function \tilde{g} with $\tilde{g}(t_1, t_1)$ not vanishing identically. We then analyse the the $2(k+1)$ -jet of g , which is its first non-zero jet and deduce from that the number of solutions of $\tilde{g}(t_1, t_2) = 0$. \square

We consider now the multi-local singularities of f_c at two lightlike points.

Theorem 3.7 *Suppose that f_{c_0} has an $A_k A_l$ -singularity at two lightlike points $p = \gamma(t_1)$ and $q = \gamma(t_2)$, with c_0 distinct from p and q .*

(i) *Suppose that p and q are on different lines of the lightcone centred at c_0 . If k and l are not both even, then the MSS is a curve with limiting tangent direction parallel to $p + q$. If k and l are both even, then the MSS is either an isolated point or a pair of curves with common limiting tangent directions parallel to $p + q$.*

(ii) *If p and q are on the same lightlike line L of the lightcone centred c_0 , then the MSS consists of the lightlike line L .*

See Figure 7.

Proof (i) We parametrise one piece of the curve by $\gamma_1(t) = (x_1 + t, -x_1 - t + \beta_1(t))$ and the second by $\gamma_2(s) = (x_2 + s, x_2 + s + \beta_2(s))$, with s, t varying independently near zero, $p = \gamma_1(0) = (x_1, -x_1)$, $q = \gamma_2(0) = (x_2, x_2)$ and $c_0 = (0, 0)$. Suppose here that $x_1 \neq 0$ and $x_2 \neq 0$, that is p and q are distinct from c_0 .

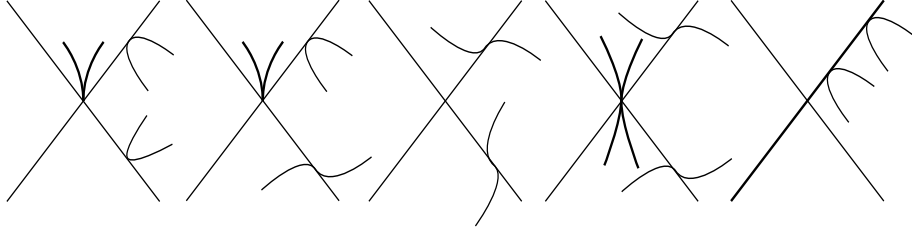


Figure 7: The *MSS* (thick), the two pieces of γ (thin) and the two lightlike tangent lines: first two figures when k and l are not both even, third and fourth when they are, and the fifth when the bi-tangency is with one branch of the lightcone.

The family of distance-squared functions is given by the bi-germ

$$\begin{cases} f_1(t, c) = -(x_1 + t - a)^2 + (x_1 + t - \beta_1(t) + b)^2 \\ f_2(s, c) = -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2 \end{cases}$$

with $c = (a, b)$ varying near $(0, 0)$. The centre c is on the *MSS* if and only if there exist s, t such that

$$\begin{aligned} \frac{\partial f_1}{\partial t}(t, c) &= 0 \\ \frac{\partial f_2}{\partial s}(s, c) &= 0 \\ f_1(t, c) &= f_2(s, c) \end{aligned}$$

equivalently,

$$a + b - b\beta_1'(t) - x_1\beta_1'(t) - \beta_1(t) - t\beta_1'(t) + \beta_1'(t)\beta_1(t) = 0 \quad (10)$$

$$a - b - b\beta_2'(s) + x_2\beta_2'(s) + \beta_2(s) + s\beta_2'(s) + \beta_2'(s)\beta_2(s) = 0 \quad (11)$$

$$-(x_1 + t - a)^2 + (x_1 + t - \beta_1(t) + b)^2 = -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2 \quad (12)$$

From (10)–(11) we get

$$b = \frac{x_2\beta_2'(s) + x_1\beta_1'(t) + \beta_2(s) + \beta_1(t) + s\beta_2'(s) + t\beta_1'(t) + \beta_2'(s)\beta_2(s) - \beta_1'(t)\beta_1(t)}{2 + \beta_1'(s) - \beta_1'(t)},$$

and substituting into (10) (or into (11)) yields a as a function in s, t .

We write $\beta_1(t) = \beta_{k+1}t^{k+1} + h.o.t.$ and $\beta_2(s) = \beta_{l+1}s^{l+1} + h.o.t$ and assign the weights $\text{weight}(s) = k$ and $\text{weight}(t) = l$ to s and t . Then

$$\begin{aligned} a &= \frac{1}{2}(x_1(k+1)\beta_{k+1}t^k - x_2(l+1)\beta_{l+1}s^l) + h.o.w. \\ b &= \frac{1}{2}(x_1(k+1)\beta_{k+1}t^k + x_2(l+1)\beta_{l+1}s^l) + h.o.w. \end{aligned}$$

where *h.o.w.* is short for “terms of higher order weights”.

Substituting in (12) the expressions for a and b as functions of (s, t) gives an equation of the form

$$g(s, t) = 0,$$

with

$$g(s, t) = x_1^2(k+1)\beta_{k+1}t^k + x_2^2(l+1)\beta_{l+1}s^l + h.o.w.$$

The function g is Newton non-degenerate, so following the same arguments in the proof of Theorem 3.6, its zero set consists of finite number of branches. The number of branches and their limiting tangent directions are determined by the solutions of the initial part $x_1^2(k+1)\beta_{k+1}t^k + x_2^2(l+1)\beta_{l+1}s^l = 0$.

Suppose that k and l are not both even. Then $g(s, t) = 0$ consists of one branch which can be parametrised, for instance when k is not even, by

$$(s, t) = (\omega^k + h.o.w, \alpha\omega^l + h.o.w), \quad (13)$$

with $\alpha^k = -x_2^2(l+1)\beta_{l+1}/(x_1^2(k+1)\beta_{k+1})$. (In particular, the curve is smooth if and only if $k = 1$ or $l = 1$, compare with [6, 14].) Substituting this into a and b gives

$$\begin{aligned} a &= \frac{x_2^{l+1}\beta_{l+1}}{x_1} (x_2 + x_1)\omega^{kl} + h.o.w. \\ b &= \frac{x_2^{l+1}\beta_{l+1}}{x_1} (x_2 - x_1)\omega^{kl} + h.o.w. \end{aligned}$$

which defines a curve in the (a, b) -plane with limiting tangent direction at the origin parallel to $(x_2 + x_1, x_2 - x_1) = p_2 + p_1$. (Observe that the curve is smooth if and only if $k = l = 1$, compare with [6, 14].)

If k and l are both even, $g(s, t) = 0$ is an isolated point if $\beta_{k+1}\beta_{l+1} > 0$ and a pair of curves with the same limiting tangent directions if $\beta_{k+1}\beta_{l+1} < 0$. The result follows by similar calculations to those for the case when k and l are not both even.

(ii) The proof is similar to that of (i). We parametrise one piece of the curve by $\gamma_1(t) = (x_1 + t, x_1 + t + \beta_1(t))$ and the second by $\gamma_2(s) = (x_2 + s, x_2 + s + \beta_2(s))$, with s, t varying independently near zero, $p = \gamma_1(0) = (x_1, x_1)$, $q = \gamma_2(0) = (x_2, x_2)$ and $c_0 = (0, 0)$. We suppose here that $x_1 \neq 0$, $x_2 \neq 0$ and $x_1 \neq x_2$.

The family of distance-squared functions is given by the bi-germ

$$\begin{cases} f_1(t, c) &= -(x_1 + t - a)^2 + (x_1 + t + \beta_1(t) - b)^2 \\ f_2(s, c) &= -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2 \end{cases}$$

with $c = (a, b)$ varying near $(0, 0)$. The centre c is on the *MSS* if and only if there exist s, t such that

$$a - b - b\beta_1'(t) + x_1\beta_1'(t) + \beta_1(t) + t\beta_1'(t) + \beta_1'(t)\beta_1(t) = 0 \quad (14)$$

$$a - b - b\beta_2'(s) + x_2\beta_2'(s) + \beta_2(s) + s\beta_2'(s) + \beta_2'(s)\beta_2(s) = 0 \quad (15)$$

$$-(x_1 + t - a)^2 + (x_1 + t + \beta_1(t) - b)^2 = -(x_2 + s - a)^2 + (x_2 + s + \beta_2(s) - b)^2 \quad (16)$$

It is clear that when $s = t = 0$, the whole line $a = b$ is a solution of the above system. Now, (14)–(15) gives

$$b = b(s, t) = \frac{x_1\beta'_1(t) - x_2\beta'_2(s) + \beta_1(t) - \beta_2(s)t\beta'_1(t) - s\beta'_2(s) + \beta'_1(t)\beta_1(t) - \beta'_2(s)\beta_2(s)}{\beta'_2(t) - \beta'_2(s)}.$$

Substituting b for $b(s, t)$ in (14) gives an expression for a as a function in (s, t) , say $a(s, t)$. Substituting in (16) a and b for $a(s, t)$ and $b(s, t)$ gives an equation in s, t . Substituting the solution of this equation (when it exists) in $b(s, t)$ gives $b(0, 0) \neq 0$. Therefore, the system of equations (14), (15) and (16) gives a unique branch of the *MSS* which is the lightlike line tangent to the two pieces of curves at p and q . \square

4 Existence of bi-tangent pairs

We consider in this section the problem of existence of bi-tangent pseudo-circles in \mathbb{R}_1^2 (for curves and surfaces in the Euclidean case see [7] and for general hypersurfaces in the Euclidean n -space see [9]).

Theorem 4.1 *Let γ be a closed simple smooth curve in \mathbb{R}_1^2 and let p be a point on γ . Then there exists a pseudo-circle or a line tangent to γ at p and at some other point q on γ distinct from p .*

Proof We consider separately the cases when p is spacelike, timelike or lightlike.

Suppose that p is a lightlike point with tangent direction $(\pm 1, 1)$. It follows from the proof of Proposition 1.2 in [14] that there is a point q on γ (in fact there are at least two such points) with tangent directions $(\mp 1, 1)$. The tangent lines to γ at p and q intersect at some point c . The lightcone $LC^*(c)$ is therefore bi-tangent to γ at p and q .

Suppose that γ is timelike at p which we take to be the origin. We follow the same setting as in [7]. By a Lorentz transformation, we write $\gamma = (x(t), y(t))$ with $\gamma(0) = p$ and $\gamma'(0) = (1, 0)$. Let $\gamma(t)$ be a point on the curve γ which is not on the x -axis (so $y(t) \neq 0$) and let $S_1^1(c(t), r(t))$ be the unique pseudo-circle that is tangent to γ at p and passes through $\gamma(t)$ (see Figure 8).

We have $c(t) = (0, r(t))$ so that

$$\langle (x(t), y(t) - r(t)), (x(t), y(t) - r(t)) \rangle = r(t)^2$$

gives

$$r(t) = \frac{-x(t)^2 + y(t)^2}{2y(t)}.$$

Note that the case when $y(t) = 0$ corresponds to the “pseudo-circle” being a straight line. We have $r'(t) = (-2x(t)y(t)x'(t) + (x(t)^2 + y(t)^2)y'(t))/(2y(t)^2)$ which vanishes if and only

$$\langle (x'(t), y'(t)), (2x(t)y(t), x(t)^2 + y(t)^2) \rangle = 0.$$

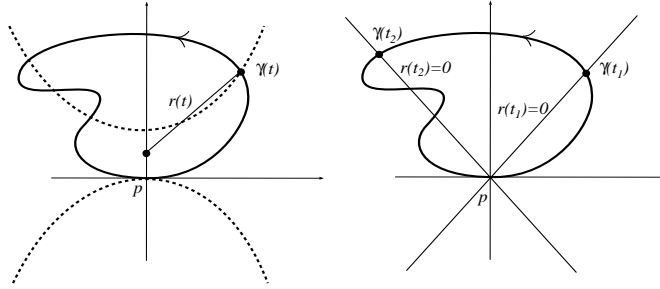


Figure 8: Bi-tangent pairs.

The above condition implies that the tangent line to the curve is pseudo-orthogonal to the Minkowski normal of the above pseudo-circle. Therefore, $r'(t) = 0$ if and only if the $S_1^1(c(t), r(t))$ is bi-tangent to γ at the origin and at $\gamma(t)$.

We consider the lightcone $LC^*(p)$ given by the two lines $(\pm s, s)$, $s \in \mathbb{R} \setminus 0$. Suppose, without loss of generality, that the interior of γ contains points in the semi-plane $y > 0$. Let $\gamma(t_1)$ and $\gamma(t_2)$ be the last points of intersection of γ with the semi-lines (s, s) and $(-s, s)$ with $s > 0$, respectively, see Figure 8(right). Here, last point of intersection means that there are no points of intersection of the curve with the semi-lines $\gamma(t_1) + (s, s)$ and $\gamma(t_2) + (-s, s)$, $s > 0$.

We orient γ so that $0 < t_1 < t_2$, and as it is a simple curve, we have $\gamma(t) \neq p$ for all $t \in [t_1, t_2]$. The function $r(t)$ is well defined at t_1 and t_2 and is smooth in the interval $[t_1, t_2]$. As $r(t_1) = r(t_2) = 0$, there exists $t_3 \in (t_1, t_2)$ for which $r'(t_3) = 0$. This implies that there exists a bi-tangent pseudo-circle to γ at $p = \gamma(0)$ and $q = \gamma(t_3)$, with $q \neq p$.

The case when γ is spacelike at p follows in a similar way to the timelike case. \square

Remark 4.2 When $p \in \gamma$ is timelike or spacelike, the proof of Theorem 4.1 shows that there exists a bi-tangent pair (p, q) with p and q belonging to different timelike or spacelike components of γ . Indeed, from the setting in the proof (for the timelike case), the tangent directions to γ at p and q are parallel to $(1, 0)$, so there must be points on γ between p and q and between q and p where the tangent line is parallel to $(\pm 1, 1)$. That is, p and q are on different components of γ .

We consider now bi-tangent pairs belonging to the same spacelike or timelike component of γ .

Proposition 4.3 *Bi-tangent pairs on the same timelike (or spacelike) component of γ can only occur on one branch of a pseudo-circle $S_1^1(c, r)$ (or $H^1(c, r)$).*

Proof Consider a timelike component C of γ , the spacelike case follows similarly. Suppose that C is tangent to one branch of a pseudo-circle $S_1^1(c, r)$ at $\gamma(t_1)$ and to the

other branch of $S_1^1(c, r)$ at $\gamma(t_2)$. Then the normal directions $\gamma(t_1) - c$ to γ at t_1 and $\gamma(t_2) - c$ to γ at t_2 are in two different spacelike components of \mathbb{R}_1^2 . Therefore, there must exist $t_3 \in (t_1, t_2)$ where the normal to C at t_3 is a lightlike direction. This is a contradiction as C is a timelike component of γ . \square

We consider the question of the existence of bi-tangent pairs on the same spacelike or timelike component of γ . If the component of γ is not strictly convex, then the following example shows that there may not exist bi-tangent pairs.

Proposition 4.4 *There are no bi-tangent pairs on the timelike or spacelike components of the curve (t, t^3) or (t^3, t) , with $t \in I = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.*

Proof Consider the case $\gamma(t) = (t, t^3)$ (the other case follows similarly). The curve γ is timelike for $t \in I$ and $\gamma(\pm\frac{1}{\sqrt{3}})$ are lightlike points. In I , γ can only be tangent to the pseudo-circles $S_1^1(c, r)$.

There exist a bi-tangent pair (t_1, t_2) on γ in I if and only if there exists $c = (a, b)$ and $r > 0$ such that the function

$$d(t) = -(t - a)^2 + (t^3 - b)^2 - r^2 = t^6 - 2bt^3 - t^2 + 2at - a^2 + b^2 - r^2$$

has a double root at t_1 and t_2 . That is,

$$d(t) = (t - t_1)^2(t - t_2)^2(t^2 + \lambda t + \mu)$$

for some λ and μ in \mathbb{R} . Equating the coefficients of t^i in both expressions of $d(t)$ yields

$$\begin{aligned} \text{Coefficient of } t^5 : & & -2t_1 - 2t_2 + \lambda & = & 0 \\ \text{Coefficient of } t^4 : & & t_1^2 + 4t_1t_2 + t_2^2 - 2\lambda(t_1 + t_2) + \mu & = & 0 \\ \text{Coefficient of } t^2 : & & -7t_1^2t_2^2 - 4t_1^3t_2 - 4t_1t_2^3 + \mu(t_1^2 + 4t_1t_2 + t_2^2) & = & -1. \end{aligned}$$

The first two equations give λ and μ as functions of (t_1, t_2) and substituting these into the third equation gives

$$3(t_1^2 + t_1t_2 + t_2^2)(t_1^2 + 3t_1t_2 + t_2^2) = -1. \quad (17)$$

Consider the function $g_{t_2}(t_1) = 3(t_1^2 + t_1t_2 + t_2^2)(t_1^2 + 3t_1t_2 + t_2^2)$, with t_2 fixed in the interval I . We have $g'_{t_2}(t_1) = 6(t_1 + 2t_2)(2t_1^2 + 2t_1t_2 + t_2^2)$, so g_{t_2} has a minimum at $t_1 = -2t_2$ with $g_{t_2}(-2t_2) = -9t_2^4$. As $t_1, t_2 \in I$, $g_{t_2}(t_1) \geq g_{t_2}(-2t_2) > -1$. Therefore, equation (17) has no solutions for $t_1, t_2 \in I$. \square

In view of Proposition 4.4, we shall restrict to timelike/spacelike components of curves γ with nowhere vanishing curvature. Such components are strictly convex curves. At lightlike points we say that γ is strictly convex if it has A_1 contact with its tangent line.

Theorem 4.5 *Let C be a spacelike or timelike component of γ defined in an open interval I and suppose that it is strictly convex on \bar{I} . Suppose that the derivative κ' of the curvature of C is not flat at any point in I . Then C has bi-tangent pairs.*

Proof As C is strictly convex, its curvature κ does not vanish on I so has constant sign on I , say positive. The curvature tends to $+\infty$ as t tends to the boundaries of I (see [14]), so it must have a global minimum on I , say at t_0 . As the curvature of C is not flat, we take k to be the least integer such that $\kappa^{(k-1)}(t_0) \neq 0$. As t_0 is a minimum of κ , k is an odd integer, and the result follows by Theorem 3.4(ii). \square

The result in Theorem 4.5 asserts the existence of bi-tangent pairs. However, we would like to know that if given any point p on the component C of γ , whether or not there exists another point q on C such that (p, q) is a bi-tangent pair. One idea is to keep track of the bi-tangent pairs born at the global minimum of the curvature t_0 in the proof of Theorem 4.5 and hope that they will reach the boundary points of I . The problem is that at an A_1A_2 -singularity of the distance-squared function one of the points of the bi-tangent pair turns back, see Figure 9. Thus, Theorem 4.5 cannot guarantee that given any point p on C there exists another q on C such that (p, q) is a bi-tangent pair.

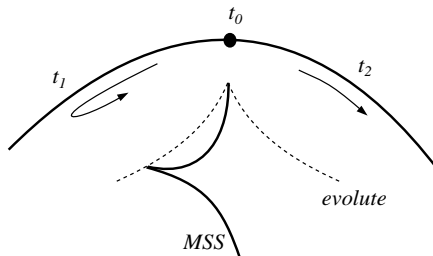


Figure 9: Turning points of bi-tangent pairs.

We need the following concept. A germ g of a smooth function is said to be finitely determined if there exists an integer k such that any other germ f with the same k -jet as g can be written as $f = g \circ h$, where h is a germ of a diffeomorphism.

Definition 4.6 *We say that a spacelike or a timelike curve γ (not necessarily connected) satisfies the property (d) if its pre-MSS is locally at any of its points the zero set of a finitely determined germ of a smooth function.*

Remark 4.7 The pre-MSS is a smooth curve at the multi-local singularities of the distance squared function on a generic curve (see [14, 6]). This is also true at the A_3 -singularities of the distance squared function. Thus, property (d) is a generic property

of plane curves. Suppose that the distance squared function f_{c_0} has a more degenerate multi-local singularity of type $A_k A_l$ at p and q , with say $l \geq k$. Take p to be the origin and parametrise the curve there by $(t, \alpha_2 t^2 + h.o.t)$. Parametrise the curve at $q = (x_0, y_0)$ by $(x_0 + s, y_0 + \beta_1 s + \beta_2 s^2 + h.o.t)$, $\beta_1 \neq 0$. The distance squared functions at p and q have the form $f_1(t, c_0) = \phi_{k+1} t^{k+1} + h.o.t$ and $f_2(s, c_0) = \psi_{l+1} s^{l+1} + h.o.t$ respectively, with $\phi_{k+1} \psi_{l+1} \neq 0$. Let $g(t, s)$ be the function whose zero set is the pre- MSS . Following the same calculations as in the proof of Theorem 3.7, and setting $\text{weight}(t) = l$ and $\text{weight}(s) = k + 1$, we show that the principal part of $g(t, s)$ (i.e., the part that determines its Newton polyhedron, see [11, 2]) is given by

$$-k\phi_{k+1}t^{k+1} + \frac{\beta_2}{\beta_1}x_0\phi_{k+1}st^k + \frac{1}{2\beta_1}y_0(l+1)\psi_l s^l. \quad (18)$$

If $l \neq k + 1$ or if $l = k + 1$ but (18) does not have a repeated root, the principal part of g is non-degenerate. It follows then that g is finitely determined so γ satisfies property (d) at the $A_k A_l$ -singularity of f_c .

Theorem 4.8 *Let C be a spacelike or timelike component of γ defined in the open interval $I = (\lambda, \mu)$ and satisfying property (d). Let c_0 be the centre of the lightcone bi-tangent to γ at λ and μ and suppose that the distance-squared function f_{c_0} has an $A_k A_l$ -singularity, with k and l both odd (i.e., γ is convex at λ and μ). Then given any point p on C , there exists a branch of a pseudo-circle that is bi-tangent to C at p and at another point q on C .*

Proof Consider the pre- MSS which consists of bi-tangent pairs (t_1, t_2) in $A = \bar{I} \times \bar{I}$, where $\bar{I} = [\lambda, \mu]$. From the parametrisation (13) of the pre- MSS , when k and l are both odd, the pre- MSS has a single branch S starting at the point (λ, μ) with the following property: for any t_1 near λ with $t_1 > \lambda$, there exists t_2 near μ with $t_2 < \mu$ such that $(t_1, t_2) \in S$. (Here the germs of curves at $\gamma(\lambda)$ and $\gamma(\mu)$ come from the same convex component C of γ , so β_{k+1} and β_{l+1} in the proof of Theorem 3.7 have the same sign, which implies that $\alpha < 0$ in (13).) Therefore, there is one smooth segment of the branch S in the interior of A near the corner (λ, μ) . We call (λ, μ) the starting point of S .

Suppose that the branch S returns to the line segment $t_2 = \mu$ in A at a point (t_1, μ) . The case $\lambda < t_1 < \mu$ is not possible as the tangent line to γ at t_1 is not lightlike (Figure 10, left). The case $t_1 = \lambda$ is also not possible because there is only one branch of S in the interior of A near (λ, μ) (see (13)). The case $t_1 = \mu$ is not possible either for the following reason. As the pre- MSS is symmetric with respect to the diagonal, $t_1 = \mu$ means that the MSS would have two branches at the point $\gamma(t_2)$. By Theorem 3.6 this is not possible as l is odd (the MSS should be locally empty at $\gamma(\mu)$).

It follows from the above considerations that the branch S does not return to the line segment $t_2 = \mu$. As the pre- MSS is a compact set, S must end somewhere in A .

Because C satisfies property (d) the end point cannot be an interior point of A , so S must cross the diagonal in A at a point which corresponds to a vertex of C , and by symmetry, ends at (μ, λ) the diagonally opposite corner in A of the starting point of S (Figure 10, right). This means that for any point $p \in C$, there exists $q \in C$, such that p, q is a bi-tangent pair. \square

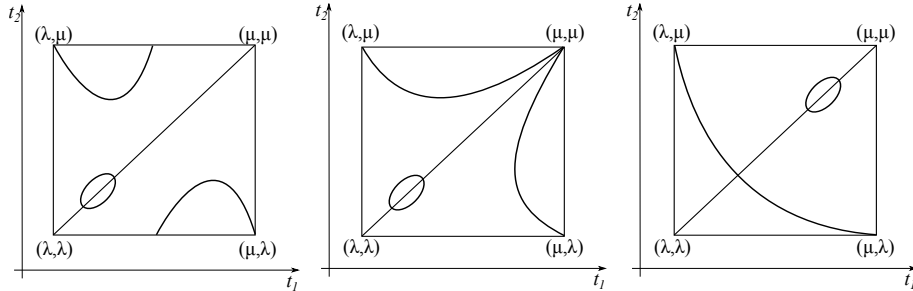


Figure 10: Impossible (left and centre) and possible (right) configurations of the pre-MSS of a timelike or spacelike component of γ . The pre-MSS can have other components such as the loop depicted in the figure.

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