Equidistants and Their Duals for Families of Plane Curves

Peter Giblin & Graham Reeve

Abstract.

We consider the local geometry of a generic 1-parameter family of smooth curves in the real plane for which one member of the family has parallel tangents at two inflexion points. We study the equidistants of this family, that is the loci of points at a fixed ratio along chords joining points with parallel tangents, as a 2-parameter family depending on the value of the fixed ratio and on the parameter in the family of curves. Codimension 2 singularities of type 'gull' arise in this way and are in general versally unfolded by the two parameters. We also calculate the family of duals of the equidistants; here it is necessary to view them as bifurcation sets of bigerms and they evolve through 'moth' and 'nib' singularities also encountered in 1-parameter families of symmetry sets in the plane. Finally we show that certain sub-families of the 2-parameter family of equidistants can be classified by reduction to a normal form.

§1. Introduction

A generic smooth, closed plane curve C will not possess two inflexion points at which the tangents are parallel, but a generic 1-parameter family of plane curves $\{C_\varepsilon\}$ can be expected to contain isolated members with this property. In a previous article [6], the authors investigated some affinely invariant constructions—equidistants and centre symmetry sets—based on such a 1-parameter family of plane curves. Besides the parameter ε in the curve family there is an additional parameter λ inherent in the equidistants: for each λ we take all chords joining pairs of points of C_ε at which the tangents are parallel and construct the locus of points at a fixed ratio $\lambda : 1 - \lambda$ along these chords.

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In [6] we were not able to consider this 2-parameter family of equidistants as a whole, but in the present article we give a method for doing this, describing the evolution of equidistants by unfoldings of singularities of maps from the plane to the plane. The most degenerate case is that of a 'gull singularity', in the language of [5], and in general this is versally unfolded by the parameters λ, ε. In addition to this we are able to analyse the inflexions, that is the *dual* structure of the equidistants, using an approach via multi-local germs of mappings reminiscent of the classification of 1-parameter families of symmetry sets in [3].

The classification of maps from the plane to the plane has another natural application, in the study of projections of smooth surfaces in $\mathbb{R}^3$ to the plane, where the critical values of the projection form the apparent contour (also called outline or profile) of the surface for a given direction of projection. Many of the same singularities cusp, swallowtail, lips, beaks, and including the gull singularity, occur in that context, and the duals exhibit the same singularities—for example the dual of a gull is also a gull. In our situation the duals have a completely different structure and are described by means of Maxwell sets of bi-germs of functions (called 'moth' and 'nib' in [3]).

All our results are local in nature and the article is organized as follows. In §2 we give the local form of the family of curves being studied, the same as in [6]. In §3 we explain our first method of studying the whole family of equidistants and show how they evolve with varying λ, ε. In §4 we study the *duals* of the equidistants, that is we capture their inflexions and bitangents, features which are not preserved by the methods of §3. In §5 we find the loci along which the geometry of the equidistants, including the inflexions, changes, and give illustrations of the results. Finally in §6 we adopt a different approach to the problem, via reduction of families to a normal form. There is an Appendix filling in some details from §6.

§2. Families of curves

Consider a family of plane curves, one member of the family having parallel but not identical tangents at inflexion points. By means of a family of affine maps of the plane to itself (compare [6]) this situation is modelled locally by two local families of curves $y = f(x, \varepsilon)$, $y = g(x, \varepsilon)$,

$$\begin{aligned}
f(x,\varepsilon) &= x^3 f_1(x,\varepsilon) \\
&= f_{30}x^3 + f_{40}x^4 + f_{31}x^3\varepsilon + f_{50}x^5 + f_{41}x^4\varepsilon + f_{32}x^3\varepsilon^2 + \dots, \\
g(x,\varepsilon) &= 1 + xg_1(\varepsilon) + x^3 g_2(x,\varepsilon) \\
(1) \qquad &= 1 + g_{11}x\varepsilon + g_{30}x^3 + g_{12}x\varepsilon^2 + g_{40}x^4 + g_{31}x^3\varepsilon + g_{13}x\varepsilon^3 + \dots,
\end{aligned}$$

where (at least) $f_{30}, g_{30}, f_{30} - g_{30}, g_{11}$ are nonzero. For all ε, the first curve has a horizontal inflexion at the origin, and the second curve has an inflexion at $(0, 1)$.

We shall consider two situations; see Figure 1.

Case 1 Here, f_{30}, g_{30} have the same sign, say positive, when we write $f_{30} = a_3^2,\ g_{30} = b_3^2$ where $a_3 > 0,\ b_3 > 0$.

Case 2 Here f_{30}, g_{30} having opposite signs, and we write $f_{30} = a_3^2,\ g_{30} = -b_3^2$ where $a_3 > 0,\ b_3 > 0$.

Note In general we do not need to assume $a_3 \neq b_3$ in what follows, even though $a_3 = b_3$ implies a greater degree of 'similarity' between the two inflexions. We shall note below when $a_3 \neq b_3$ is required.

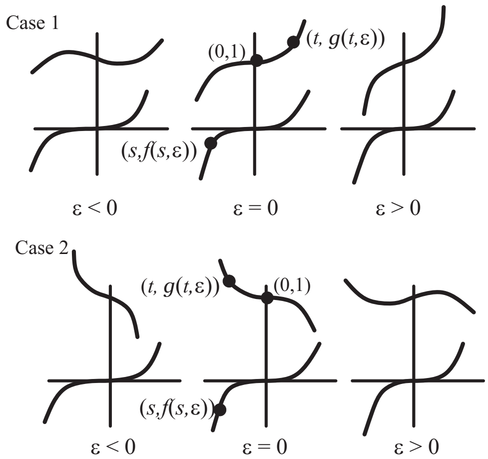

Fig. 1. Schematic representation of the two cases considered here. Case 1 ($f_{30}g_{30} > 0$ in (1)) has the inflexions 'oriented the same way' or 'of the same sign' and Case 2 ($f_{30}g_{30} < 0$) 'oriented opposite ways' or 'of opposite signs'. The sign given for ε assumes (without loss of generality) that $g_{11} > 0$ in the notation of (1), that is the upper inflexional tangent turns anticlockwise as ε increases. For Case 2 there are no pairs of parallel tangents at parameter points s, t for $\varepsilon < 0$, and for $\varepsilon > 0$ the ranges of values of s, t providing parallel tangents are bounded.

We are interested in the *equidistants* which are the points which are at a fixed ratio along the chord joining a pair $(s, f(s, \varepsilon))$ and $(t, g(t, \varepsilon))$ at which the tangents to the two curves are parallel (we do not include pairs where both belong to the same local curve). Thus the equidistant, for a fixed λ, consists of points

$$(1-\lambda)(s, f(s,\varepsilon)) + \lambda(t, g(t,\varepsilon)),$$

subject to the parallel tangency condition.

Definition 2.1. (i) When considering values of λ close to some fixed $\lambda_0 \neq 0, 1$ we write $\lambda = \lambda_0 + \alpha$.
(ii) For Case 1 it will be necessary to separate out two *special values* of λ_0, namely $\lambda_0 = \dfrac{b_3}{b_3 \pm a_3}$. (For the $-$ sign we require $a_3 \neq b_3$.) These were also discussed in [6]. For Case 2 there are no special values of λ_0.

§3. Equidistants

For any pair of smooth plane curves varying in a 1-parameter family, $\gamma_1(s,\varepsilon)$ and $\gamma_2(t,\varepsilon)$, we can consider the "λ-point map", defined by

$$(s, t, \lambda, \varepsilon) \mapsto (1-\lambda)\gamma_1(s,\varepsilon) + \lambda\gamma_2(t,\varepsilon) \in \mathbb{R}^2.$$

For fixed λ, ε the critical set consists of points $(s, t, \lambda, \varepsilon)$ where the tangent lines to the two curves at $\gamma_1(s,\varepsilon)$ and $\gamma_2(t,\varepsilon)$ are parallel. Thus for fixed λ, ε the discriminant of this mapping, that is the set of its critical values, is exactly the equidistant for those values of λ, ε. Extending the map to

$$(s, t, \lambda, \varepsilon) \mapsto ((1-\lambda)\gamma_1(s,\varepsilon) + \lambda\gamma_2(t,\varepsilon), \lambda, \varepsilon) \in \mathbb{R}^4,$$

the discriminant is the union of all equidistants of all members of the family.

For our family this becomes

$$(s, t, \lambda, \varepsilon) \mapsto ((1-\lambda)s + \lambda t,\ (1-\lambda)f(s,\varepsilon) + \lambda g(t,\varepsilon))\,. \tag{2}$$

In order to compare this more easily with standard forms we write $u = (1-\lambda)s + \lambda t$ and solve for t:

$$t = \frac{u - (1-\lambda)s}{\lambda},$$

where we shall avoid working near $\lambda = 0$. We shall also write $\lambda = \lambda_0 + \alpha$ for a fixed $\lambda_0 \neq 0, 1$. Then the above family will be written as follows:

$$\mathcal{H}(s, u, \alpha, \varepsilon) = (u,\ H(s, u, \alpha, \varepsilon)), \tag{3}$$

where H is a real-valued function. We shall use the following abbreviations:

$$H_0(s,u) = H(s,u,0,0); \quad \mathcal{H}_0(s,u) = \mathcal{H}(s,u,0,0) = (u, H_0(s,u)).$$

We proceed to examine the map $\mathcal{H}_0 : \mathbb{R}^2, (0,0) \to \mathbb{R}^2$ and its unfolding $\mathcal{H} : \mathbb{R}^2 \times \mathbb{R}^2, (0,0,0,0) \to \mathbb{R}^2$ and prove the following proposition, where we assume as always that $\lambda \neq 0, 1$.

Proposition 3.1. *For Case 1,*
(i) if $\lambda_0 \neq \dfrac{b_3}{b_3 \pm a_3}$*, then the map* $\mathcal{H}_0$ *has the type 'beaks', and is versally unfolded by the* ε *parameter alone;*
(ii) if $\lambda_0 = \dfrac{b_3}{b_3 \pm a_3}$ *(for the* $-$ *sign we require* $a_3 \neq b_3$*) then the map* $\mathcal{H}_0$ *has type 'gull'*[1] *and is versally unfolded by the parameters* α, ε *for generic values of the coefficients. The precise conditions are given below in* (6)*.*
For Case 2, and any λ*,*
(iii) the map $\mathcal{H}_0$ *has the type 'lips', and is versally unfolded by the* ε *parameter alone.*

Proof (i) Then $H_0(s,u) = \lambda_0 + c_{30}s^3 + c_{21}s^2u + c_{12}su^2 + c_{03}u^3 +$ higher terms, where

$$c_{30} = -\frac{(1-\lambda_0)(\lambda_0(b_3+a_3)-b_3)(\lambda_0(b_3-a_3)-b_3)}{\lambda_0^2} \neq 0.$$

It is then easy to remove the s^2u term by substituting $s = s_1 + ku$ where $k = -c_{21}/(3c_{30})$. Using $\mathcal{A}$-equivalence on the map $\mathcal{H}_0$, the term in u^3 can be removed too. The result in the present situation is

$$\begin{aligned} &\mathcal{H}_0(s_1,u) \\ = \quad &\left(u,\ \lambda_0 + c_{30}s_1^3 + \frac{3(1-\lambda_0)a_3^2b_3^2}{(\lambda_0(b_3-a_3)-b_3)(\lambda_0(b_3+a_3)-b_3)}s_1u^2 + \dots,\right) \end{aligned}$$

where both coefficients are nonzero. This is essentially a normal form for a (3-$\mathcal{A}$-determined) *lips/beaks singularity*, where beaks occurs if and only

[1]See [5]. In [8], Clint McCrory describes three types of 'gulls' [plural] singularity which arise from local projections of surfaces to the plane, one of 'elliptic type' and two of 'hyperbolic type'. As local maps from the plane to the plane these are all $\mathcal{A}$-equivalent but the patterns of inflexions differ. We shall see in the next section that the pattern of inflexions on the equidistants is quite different from this.

if the coefficients of s_1^3 and s_1u^2 have opposite sign. But the product of these coefficients is $-\dfrac{3(1-\lambda_0)^2a_3^2b_3^2}{\lambda_0^2}$, which is always negative. Hence the mapping $\mathcal{H}_0$ has a beaks singularity at $s_1 = u = 0$. (The lips singularity occurs for Case 2, for any value of λ.)

The function $H(s_1, u, \alpha, \varepsilon)$ when the same reduction is made has the form

(4)

$$\lambda_0 + \alpha - (1-\lambda_0)g_{11}s_1\varepsilon - \frac{a_3^2\lambda_0^2 g_{11}}{(\lambda_0(b_3-a_3)-b_3)(\lambda_0(b_3+a_3)-b_3)}u\varepsilon + \ldots.$$

This means that the 'initial speed' $\dfrac{\partial H}{\partial \alpha}$ provides, up to this order, only the constant 1, while the 'initial speed' $\dfrac{\partial H}{\partial \varepsilon}$ provides (since $g_{11} \neq 0$) the term s_1 which is needed for a versal unfolding of the beaks singularity. Thus changing λ to values close to λ_0 provides only a trivial unfolding but changing ε to values close to 0 does give a versal unfolding of the beaks singularity, as observed in [6].

(ii) We take the case $\lambda_0 = \dfrac{b_3}{b_3 - a_3}$.

Remark 3.2. Clearly this requires $a_3 \neq b_3$. However all the corresponding calculations for the other special value $\lambda_0 = \dfrac{b_3}{b_3 + a_3}$ are obtained by the formal substitution of $-b_3$ for b_3 throughout, and at that special value it is permissible to have $a_3 = b_3$ in which case $\lambda_0 = \frac{1}{2}$.

With $\lambda_0 = b_3/(b_3 - a_3)$ the s^3 term is absent from H_0 and there is a standard technique (see for example [9]) for reducing the 3-jet of H_0 to a multiple of s^2u, using $\mathcal{A}$-equivalence on $\mathcal{H}_0$. We can always remove powers of u using a left-equivalence; we then replace s by $s_1 - c_{12}u/(2c_{21})$. This gives the 4-jet up to $\mathcal{A}$-equivalence

$$j^4H_0(s_1, u) = \lambda_0 + c_{21}s_1^2u + c_{40}s_1^4 + c_{31}s_1^3u + c_{22}s_1^2u^2 + c_{13}s_1u^3,$$

where

$$c_{21} = 3a_3^2 \neq 0, \quad c_{40} = \frac{a_3(a_3^3g_{40} - b_3^3f_{40})}{b_3^3(b_3 - a_3)}$$

and for certain coefficients c_{31}, c_{22}, c_{13}. We therefore assume that the coefficient c_{40} of s_1^4 is nonzero (compare [6, Prop. 4.1]). A second substitution $s_2 = s_1 + \frac{1}{6a_3^2}(c_{31}s_1^2 + c_{22}s_1u + c_{13}u^2)$ (which is then solved for s_1 as a function of s_2, u) removes the degree 4 terms other than s_1^4 and the 5-jet of $H_0(s_2, u))$ becomes

$$\lambda_0 + 3a_3^2 s_2^2 u + c_{40} s_2^4 + c_{50} s_2^5 + \text{ other degree 5 terms}$$

where

$$c_{50} = \frac{4b_3^6 f_{40}^2 - 4a_3^6 g_{40}^2 + 3a_3^6 b_3^2 g_{50} - 3a_3^2 b_3^6 f_{50}}{3a_3 b_3^6 (b_3 - a_3)}. \tag{5}$$

The condition that c_{50} is nonzero also arose in [6, Prop. 4.2] as the additional condition that a function on a swallowtail surface yielded the transition on equidistants at a special point. We shall now assume that c_{40} and c_{50} are both nonzero. The conditions $c_{40} \neq 0$, $c_{50} \neq 0$ can be written in the more suggestive forms (as in [6])

$$\frac{f_{40}}{a_3^3} \neq \frac{g_{40}}{b_3^3}, \quad \frac{4f_{40}^2}{a_3^6} - \frac{3f_{50}}{a_3^4} \neq \frac{4g_{40}^2}{b_3^6} - \frac{3g_{50}}{b_3^4} \tag{6}$$

in which the two curves are separated on the two sides.

A further substitution $s_3 = s_3(s_2, u)$ of a similar kind removes the degree 5 terms other than s_2^5 but does not affect c_{40} or c_{50}, resulting in the normal form up to $\mathcal{A}$-equivalence

$$\mathcal{H}_0(s_3, u) = (u,\ \lambda_0 + c_{21} s_3^2 u + c_{40} s_3^4 + c_{50} s_3^5). \tag{7}$$

All three coefficients c_{ij} are nonzero, so that this jet is 5-$\mathcal{A}$-determined and occurs in Rieger's list [9, Table 1] as 11_5, and is also the 'gull singularity' of [5].

Applying the same transformations to the function H we obtain

$$\lambda_0 + \alpha + \frac{a_3 g_{11}}{b_3 - a_3} s_3 \varepsilon + \ldots + \frac{2a_3^2(b_3 - a_3)}{b_3} s_3^3 \alpha + \ldots + k s_3^3 \varepsilon + \ldots,$$

where the terms exhibited are the significant ones in the initial speeds $(0, \partial H/\partial \alpha)$ and $(0, \partial H/\partial \varepsilon)$, and k is an expression involving the coefficients of f, g encountered so far and in addition f_{31} and g_{31}. These terms guarantee that unfolding terms $(0, s_3)$ and $(0, s_3^3)$ are provided by the initial speeds and since in the normal form (7) other terms in s_3, u up to degee 5 are in the extended tangent space of $\mathcal{H}_0$, it follows that α and ε versally unfold the 'gull' singularity of $\mathcal{H}_0$ at $s_3 = u = 0$.

The proof of (iii) is similar to (i) except that there are no special values of λ here. □

§4. Duals of the equidistants

4.1. Inflexions

We should like to study the duals of the equidistants in order to decide the pattern of inflexions (not preserved by $\mathcal{A}$-equivalence). As a start we derive here the condition for an equidistant to have an inflexion. The following discussion applies both to Case 1 and to Case 2.

An equidistant has the form, for fixed λ and ε,

$$\{((1-\lambda)s+\lambda t, \quad (1-\lambda)f(s,\varepsilon)+\lambda g(t,\varepsilon))\},$$

subject to the parallel tangent condition $f_s = g_t$. For the purpose of calculation assume that the condition $f_s = g_t$ is solved as $t = T(s)$ so that $f_s(s,\varepsilon) \equiv g_t(T(s),\varepsilon)$. Similar arguments apply if s is a function of t and one or the other holds except for $f_{ss} = g_{tt} = 0$: inflexions on both curves, which happens for parallel tangents only at $s = t = \varepsilon = 0$. Note that the tangent to the equidistant is parallel to the tangents to the two given curves. In fact, for fixed λ and ε, writing

$$\delta(s) = ((1-\lambda)s+\lambda t, \quad (1-\lambda)f(s,\varepsilon)+\lambda g(T(s),\varepsilon))$$

the condition for an inflexion is that $\delta'(s)$ and $\delta''(s)$ are parallel vectors. The derivatives are

$$\delta' = ((1-\lambda)+\lambda T')(1, f_s) \text{ and } \delta'' = (\lambda T'', (1-\lambda)f_{ss}+\lambda g_{tt}T'^2+g_t T'').$$

Computing T' and T'' from $f_s(s,\varepsilon) \equiv g_t(T(s),\varepsilon)$ we find $T' = f_{ss}/g_{tt}$ and $T'' = (f_{sss}g_{tt}^2 - g_{ttt}f_{ss}^2)/g_{tt}^3$. Finally substituting these into the condition for δ' and δ'' to be parallel, and clearing denominators, gives the following condition (besides $f_s = g_t$):

$$((1-\lambda)g_{tt}+\lambda f_{ss})f_{ss}g_{tt} = 0. \tag{8}$$

However the bracket is in fact exactly the condition for the equidistant to be *singular*, that is $\delta' = 0$ so the condition reduces to $f_{ss} = 0$ or $g_{tt} = 0$. (Both occur simultaneously only for $s = t = \varepsilon = 0$.) Extending the calculation we find that the inflexion is ordinary (δ''' not parallel to δ') if and only if the corresponding derivative f_{sss} or g_{ttt} is nonzero, and this will be the case for small values of s, t, ε since a_3 and b_3 are nonzero. We deduce the following.

Proposition 4.1. *The equidistant has an inflexion at the (nonsingular) point corresponding to (s,t) where the tangents are parallel if and only if $f_{ss} = 0$ or $g_{tt} = 0$, that is one of the two curves has an inflexion. In our case this means that, for any ε, inflexions occur exactly for $s = 0$ and for $t = 0$ and all these inflexions are ordinary inflexions.* □

Corollary 4.2. *Away from any singular points of the equidistant, for Case 1 (inflexions 'facing the same way') there are exactly two inflexions on the equidistant, both ordinary, and for Case 2 (inflexions 'facing opposite ways') there are four.*

Also, the tangent to an equidistant at an inflexion point will either be parallel to the x-axis or to the line $y = g_{11}x$.

The second statement of the corollary follows because for a pair of points $\mathbf{p} = (s, f(s, \varepsilon))$ and $\mathbf{q} = (t, g(t, \varepsilon))$ with parallel tangents, the tangent at the corresponding point $(1 - \lambda)\mathbf{p} + \lambda\mathbf{q}$ to any equidistant is parallel to those tangents. The tangent at the the origin $s = 0$ to the curve $y = f(x, \varepsilon)$ is horizontal and the tangent at $(0, 1)$ $(t = 0)$ to the curve $y = g(x, \varepsilon)$ has slope g_{11}. □

4.2. The family of duals

We now explain how to study the duals of the equidistants by means of a suitable map. For fixed ε and $\lambda \neq 0, 1$ let

$$F(s, t, u, v) = ((1 - \lambda)(s, f(s, \varepsilon)) + \lambda(t, g(t, \varepsilon))) \cdot (u, 1) - v, \tag{9}$$

where $\cdot$ is the euclidean scalar product of vectors. Then $F_s = 0$ means that the tangent to the curve $\{(s, f(s, \varepsilon))\}$ is perpendicular to $(u, 1)$ and $F_t = 0$ similarly for the curve $\{(t, g(t, \varepsilon))\}$, while $F = 0$ means that $v = \mathbf{p} \cdot (u, 1)$ where $\mathbf{p}$ is the equidistant point for the value λ corresponding to parameter values s, t. Since the tangent to the equidistant has the equation $(\mathbf{x} - \mathbf{p}) \cdot (u, 1) = 0$ we can use (u, v) to parametrize the *dual* of the equidistant. For fixed λ and ε the set of points

$$\{(u, v) : \exists s, t \text{ such that } F = F_s = F_t = 0\}$$

is the dual of the equidistant for that λ and ε. This is the discriminant set of a family of functions of two variables s, t. However this description is not quite satisfactory since $F(s, t, 0, 0)$ has type $D_4^{\pm}$ and hence λ and ε cannot versally unfold the singularity.

To get round this problem we split F into two parts, $F = (F_1, F_2)$ and regard it as a *bigerm*, of singularity type A_2^2 at $u = v = \varepsilon = 0$ (and any $\lambda \neq 0, 1$), as follows. Let, again for fixed λ,

$$F_1(s, u, v, \varepsilon) = (1-\lambda)su + (1-\lambda)f(s, \varepsilon), \quad F_2(t, u, v, \varepsilon) = -\lambda tu - \lambda g(t, \varepsilon) + v.$$

Thus, $F_1(s, 0, 0, 0) = a_3^2 s^3 +$ higher terms, and $F_2(t, 0, 0, 0) = \pm b_3^2 t^3 +$ higher terms, where the sign is $+$ for Case 1 and $-$ for Case 2. All these are of type A_2 since a_3 and b_3 are nonzero. We have the following.

Proposition 4.3. *For fixed* λ, ε *the dual of the equidistant is the* levels bifurcation set *or* Maxwell set

$$\mathcal{B} = \{(u,v) : \exists s,t \text{ such that } F_{1s} = F_{2t} = 0, F_1 = F_2\}.$$

Furthermore, for any $\lambda \neq 0,1$*, the parameters* u, v, ε *in* F *give a multiversal unfolding of the* A_2^2 *singularity of this bigerm at* $(s,t) = (0,0)$.

Note that in our situation we do not need to specify $s \neq t$ since they are parameters on two separate curve pieces.

Proof of the last statement. The 'initial speeds', evaluated at $u = v = \varepsilon = 0$, are

$$F_u = ((1-\lambda)s, -\lambda t),\ F_v = (0,1) \text{ and } F_\varepsilon = ((1-\lambda)O(s^3), -\lambda g_{11}(t+O(t^3))),$$

which give the requisite terms $(s,0),(0,1)$ (or $(1,0)$) and $(0,t)$ for a multiversal unfolding of the A_2^2 singularity, since $g_{11} \neq 0$. □

This result implies that, for any λ, including special values, the levels bifurcation set $\mathcal{B}$ is locally diffeomorphic to the levels bifurcation set $\mathcal{B}_G$ of the standard multiversal unfolding of an A_2^2 singularity, namely (see [3]):

$$G_1(x,p,q,r) = x^3 + px, \quad G_2(y,p,q,r) = y^3 + qy + r,$$

$$\begin{aligned} \text{with } \mathcal{B}_G &= \{(p,q,r) : G_{1x} = G_{2y} = G_1 - G_2 = 0\} \\ &= \{(-3x^2, -3y^2, 2(x^3 - y^3))\}. \end{aligned}$$

To understand the evolution of the dual equidistants for a fixed λ and ε passing through 0 we therefore need to identify the function ε on $\mathcal{B}$. For a stable function the evolution will be, up to local diffeomorphism, a standard 'nib' or 'moth' transition from [3], and this occurs when the plane $\varepsilon = 0$ is transverse to the limiting singular strata of the surface $\mathcal{B}$, namely the limiting tangent lines to the cuspidal edges and the double curve. The 'nib' is illustrated in Figure 3 along the ε-axis and the 'moth' is the 4-cusped curve on the right of Figure 4, which shrinks to a point and disappears in the transition.

It is clear that we can solve the equations $F_{1s} = F_{2t} = 0, F_1 = F_2$ for u and v, and the remaining equation is then simply $f_s(s,\varepsilon) = g_t(t,\varepsilon)$ which is the condition for parallel tangents of the original two inflexional curves $y = f(x,\varepsilon)$ and $y = g(x,\varepsilon)$. This equation is solved locally by say $\varepsilon = E(s,t)$, such a smooth solution being guaranteed by $g_{11} \neq 0$. The lowest terms of (u,v,ε) as functions of s and t, say $(U(s,t), V(s,t), E(s,t))$, are as follows, for any λ:

$(U, V, E) =$
(10)
$$\left(-3a_3^2 s^2 + \ldots,\ \lambda + 2a_3^2(\lambda - 1)s^3 \mp 2b_3^2 \lambda t^3 + \ldots,\ \frac{3a_3^2}{g_{11}} s^2 \pm \frac{3b_3^2}{g_{11}} t^2 + \ldots \right).$$

Here the upper sign is for Case 1 and the lower sign for Case 2. Note that $U(0, t) \equiv 0$ since $U = -f_s(s, \varepsilon)$ before we substitute for ε.

The cuspidal edges on $\mathcal{B}$ correspond to inflexions on the equidistants and these occur for $s = 0$ and $t = 0$ only (see Proposition 4.1). Putting $s = 0$ or $t = 0$ in (10) then gives the limiting tangent vectors to the cuspidal edges on $\mathcal{B}$ as $(0, 0, 1)$ and $(1, 0, 1)$, both of which are transverse to the plane $\varepsilon = 0$. We therefore need to examine the limiting tangent vector to the double curve on $\mathcal{B}$. Calculation shows that the relation between s and t on the double curve of $\mathcal{B}$ is $D(s, t) = 0$ where

$$D(s, t) = a_3^2(1 - \lambda)s^3 \pm b_3^2 \lambda t^3 + \ldots,$$

so that $t = ks + \ldots$ where $k^3 = \pm a_3^2(\lambda - 1)/b_3^2\lambda$. The tangent vector to the image of this curve on $\mathcal{B}$ then has the form $(ps^3 + \ldots,\ qs^5 + \ldots, \mp 18a_3^2 b_3^2(\lambda k + 1 - \lambda)s^3/g_{11} + \ldots)$, where $p \neq 0$. The limit as $s \to 0$ is transverse to the plane $\varepsilon = 0$ unless the coefficient of s^3 in the third component is 0. This is equivalent to

$$\left(\frac{\lambda - 1}{\lambda}\right)^2 = \pm \frac{a_3^2}{b_3^2}; \text{ for the upper sign } + \text{ this is } \lambda = \frac{b_3}{b_3 \pm a_3}.$$

For the upper sign (Case 1), these are the special values of λ. For the lower sign (Case 2) there are no special values of λ. We deduce the following.

Proposition 4.4. *For Case 1 ($f_{30} = a_3^2, g_{30} = b_3^2$ in (1)) the function ε on the set $\mathcal{B}$ is stable provided λ is not one of the special values $b_3/(b_3 \pm a_3)$, and the transition on dual equidistants as ε passes through 0 is then a 'nib' transition.*

For Case 2 ($f_{30} = a_3^2, g_{30} = -b_3^2$ in (1)) , the function ε on $\mathcal{B}$ is always stable and the transition on the dual equidistants as ε passes through 0 is a 'moth' transition.

Remark 4.5. At a special value of λ in Case 1 we can examine the situation less formally by direct calculation. The transition is almost identical to a nib transition, except that for $\varepsilon = 0$ the two cusps with a common tangent are no longer both ordinary cusps. The outer one is ordinary and the inner one is rhamphoid. In terms of the standard A_2^2 surface $\mathcal{B}_G$, parametrized by $(x, y) \mapsto (u, v, w) = (x^2, y^2, x^3 - y^3)$, a function whose level sets model the transition at a special value of λ is $u - v + w^2$, while stable functions which model the nib/moth are respectively $2u - v$ and $u + v$. There is an additional condition for the non-stable function ε to be equivalent to $u - v + w^2$ and that is the second condition of (6) above.

§5. Equidistants and duals together

Our aim in this section is to describe the equidistants and their duals simultaneously, that is to include inflexions in the 'clock diagram' of the equidistants, in the (ε, α)-plane where $\lambda = \lambda_0 + \alpha$ for a fixed λ_0. The most interesting situation is Case 1, with λ_0 a special value, and we shall start with that.

Let $f_{30} = a_3^2,\ g_{30} = b_3^2$ in (1) (Case 1) and $\lambda = b_3/(b_3 - a_3) + \alpha$, and consider a neighbourhood of the origin in the (ε, α)-plane. (The calculations for the other special value $b_3/(b_3 + a_3)$ are similar.) There are some loci in the (ε, α)-plane which help us to understand the geometrical structure of the equidistants.

- (S) The set of points (ε, α) for which the equidistant has a swallowtail singularity (the dual a double inflexion or undulation). Crossing this locus, the number of cusps on the equidistant (inflexions on the dual) changes by 2, and the number of self-intersections on the equidistant (double tangents on the dual) changes by 1.
- (T) The set of points (ε, α) for which the equidistant, and hence also the dual, has a self-tangency or tacnode. Crossing this locus the number of self-intersections of the equidistant, or of the dual, changes by 2.

The calculations to identify these loci are straightforward but tedious and we state the results, as follows. We shall assume without loss of generality that $g_{11} > 0$.

Proposition 5.1. *Assume that the conditions of* (6) *both hold.*

(i) *The swallowtail locus* S *has the form*

$$\varepsilon = \frac{a_3^3 b_3^3 (a_3 - b_3)^6}{2 g_{11} (a_3^3 g_{40} - b_3^3 f_{40})^2} \alpha^3 + \dots .$$

(ii) *The self-tangency locus* T *has the form*

$$\varepsilon = \frac{3 a_3^4 b_3^4 (a_3 - b_3)^4}{g_{11} B} \alpha^2 + \dots ,$$

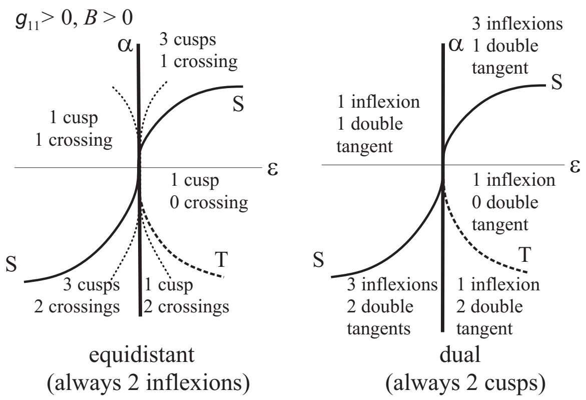

Fig. 2. Case 1, $\lambda = b_3/(b_3 - a_3) + \alpha$. The swallowtail (S) and self-tangency (T) loci for the equidistants, as in Proposition 5.1, showing how the number of self-intersections and the number of cusps on the equidistant changes around the (ε, α) diagram. The thin dashed lines in the left-hand diagram represent the coincidence of a self-intersection and an inflexion, as in Proposition 5.2. The quantity B is given by (11). If $B < 0$ then the figure is reflected in the origin, so that S remains essentially unaltered and T moves to the second quadrant.

where

$$B = 4b_3^6 f_{40}^2 - 4a_3^6 g_{40}^2 + 3a_3^6 b_3^2 g_{50} - 3a_3^2 b_3^6 f_{50} \tag{11}$$

(the same as the numerator of c_{50} *in* (5)*),* ε *has the sign of* B *and* α *has the sign of* $-B$*.*
(iii) *In addition, crossing the* α *axis,* $\varepsilon = 0$*, where beaks transitions occur on the equidistant, the number of cusps changes by two but although there is a self-tangency on the equidistant the number of self-intersections does not change.* □

These loci are illustrated in Figure 2 and the equidistants and duals themselves in Figures 3, 4.

There are two other loci in the (ε, α)-plane, where $\lambda = \lambda_0 + \alpha$ and λ_0 is a special value, which affect the configuration of the equidistants in a minor way. Firstly, an inflexion, which always corresponds to $s = 0$ or $t = 0$, can occur at the same place as a self-intersection on the equidistant. Crossing this locus an inflexion migrates from one sides

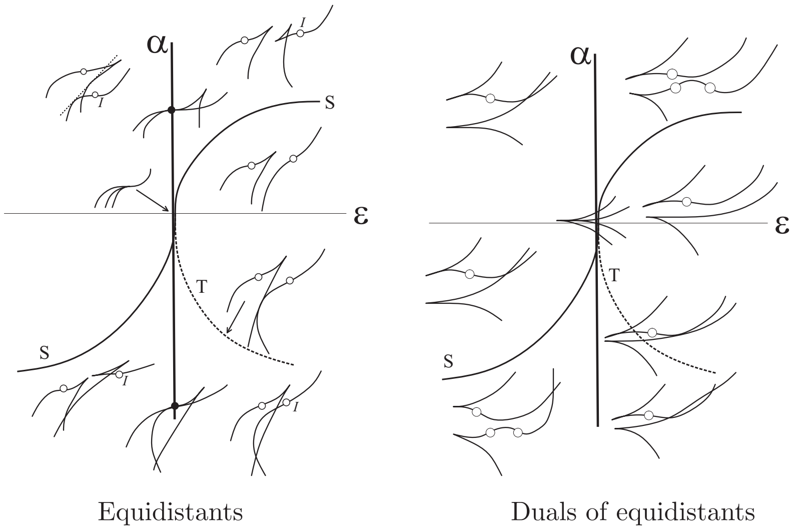

Fig. 3. (As with Figure 2 we take $g_{11} > 0, B > 0$, B given by (11).) Left: for Case 1, $\lambda = b_3/(b_3 - a_3) + \alpha$, α and ε small, this shows the evolution of the equidistants of the family of curves studied in this article. Inflexions are marked by a circle and those labelled I migrate across the nearby self-intersection as $\varepsilon \to 0$. For $\varepsilon < 0$ the inflexions on the equidistants are horizontal and for $\varepsilon > 0$ they are parallel and of positive slope g_{11} (see Corollary 4.2). One bitangent line is indicated at top left. The swallowtail locus S and self-tangency locus T are as in Figure 2. Right: a necessarily more schematic indication of the duals, since there is no canonical coordinate system in the dual plane. The transition along the ε axis is called a 'nib' in [3].

of a self-intersection to the other. It occurs only for Case 1 (there are no special values of λ for Case 2) and is given by (i) in the following proposition. Secondly, and also for Case 1, the same event can occur on the dual equidistant, which means that, on the equidistant itself, there is a bitangent line which, for one tangency, is the (limiting) tangent at a cusp. Crossing this locus a bitangent line migrates through a cuspidal tangent on the equidistant. See (ii) below.

Proposition 5.2. *Suppose as before that* $a_3^3 g_{40} - b_3^3 f_{40} \neq 0$. *Then* (i) *the locus in the* (ε, α) *plane where an inflexion and a self-intersection* *coincide on the equidistant has the form*

$$\varepsilon = \pm \frac{a_3^2 b_3^2 (a_3 - b_3)^8}{3 g_{11} (a_3^3 g_{40} - b_3^3 f_{40})^2} \alpha^4 + \ldots,$$

where the sign is $+$ *for inflexions corresponding to* $t = 0$ *and* $-$ *for inflexions corresponding to* $s = 0$.
(ii) *the locus in the* (ε, α) *plane where the dual has an inflexion coinciding with a self-intersection has the form*

$$\varepsilon = -\frac{2 a_3^3 b_3^3 (a_3 - b_3)^6}{g_{11} (a_3^3 g_{40} - b_3^3 f_{40})^2} \alpha^3 + \ldots.$$

Assuming as usual that $g_{11} > 0$, *this locus lies locally in the second and fourth quadrants of the* (ε, α) *plane where* ε *and* α *have opposite signs.* $\square$

The migration of inflexions on the equidistant is noted on Figure 3. We do not attempt to include the second migration (ii) on the same figure but an idea of what is happening on the equidistant itself is in Figure 5.

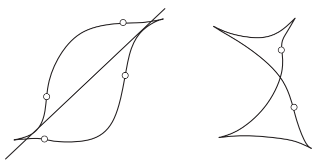

Fig. 4. Case 2, the equidistant (left, a lips) and dual (right) for $\varepsilon > 0$, with a bitangent line indicated corresponding to the self-intersection on the dual. The dual is called a 'moth' as in [3] and it disappears as $\varepsilon \to 0$. Note that this lips necessarily has four inflexions, as in Corollary 4.2.

§6. Reduction to a normal form preserving the ε-fibration

In Proposition 3.1 it was shown that the family $\mathcal{H}$ given by (3) is $\mathcal{A}$-equivalent to a versal unfolding of a lips, beaks or gull singularity. In this section the same family is considered using a finer notion of equivalence which we call (α, ε)-$\mathcal{A}$ equivalence. As before, $\lambda = \lambda_0 + \alpha$ and λ_0 is either a general value or a special value, but never 0 or 1. Restricting

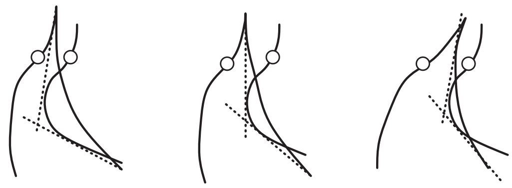

Fig. 5. Migration of an inflexion on the *dual* equidistant across a self-intersection produces this change on the equidistant, where the configuration is from the bottom right of the equidistants diagram in Figure 3. A similar change occurs in the top left but with a single bitangent line.

the permissible changes in the parameters so that ε can be replaced by a function in ε only (not involving α) preserves the fibration over ε. In particular, this preserves how the families of equidistants evolve as α varies for fixed values of ε near zero. It does not, however, preserve the families of equidistants as ε varies, for a fixed α, such as $\alpha = 0$. The advantage of the method here is that it is possible to reduce to a *normal form*.

Definition 6.1. *Two germs of families* $\mathcal{H}_i : \mathbb{R}^2 \times \mathbb{R}^2, \mathbf{0} \to \mathbb{R}^2,\ i = 1, 2$*, of the variables* s, u *and with parameters* α, ε *are called* (α, ε)-$\mathcal{A}$ *equivalent if there exists a diffeomorphism germs* $\theta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$*, of the form*

$$\theta : (s, u, \alpha, \varepsilon) \mapsto (\theta_1(s, u, \alpha, \varepsilon), \theta_2(s, u, \alpha, \varepsilon), A(\alpha, \varepsilon), E(\varepsilon))$$

and a diffeomorphism $\phi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$*, of the form* $(\phi_1, \phi_2) \times \mathrm{id}$*, which is the identity on the last two coordinates, such that* $\phi \circ (\mathcal{H}_1 \times \mathrm{id}) = (\mathcal{H}_2 \times \mathrm{id}) \circ \theta + c(\alpha, \varepsilon)$ *for a smooth germ* $c(\alpha, \varepsilon)$*. (Here* $\mathcal{H}_i \times \mathrm{id}$ *is the identity on the last two coordinates* (α, ε)*.)*

$$\begin{array}{ccc} \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\mathcal{H}_1 \times \mathrm{id}} & \mathbb{R}^2 \times \mathbb{R}^2 \\ \downarrow \theta & & \downarrow \phi \\ \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\mathcal{H}_2 \times \mathrm{id}} & \mathbb{R}^2 \times \mathbb{R}^2 \end{array}$$

In other words,

$$\begin{aligned} & (\phi_1(\mathcal{H}_1(s,u,\alpha,\varepsilon),\alpha,\varepsilon),\phi_2(\mathcal{H}_1(s,u,\alpha,\varepsilon))) \\ = \; & \mathcal{H}_2(\theta_1(s,u,\alpha,\varepsilon),\theta_2(s,u,\alpha,\varepsilon),A(\alpha,\varepsilon),E(\varepsilon))+c'(\alpha,\varepsilon) \end{aligned}$$

where c' is the first two components of c. (The last two components must be of the form $(\alpha - A, \varepsilon - E)$.) The key requirement is of course that the equivalence preserves, up to local diffeomorphism, the critical point set and its image, the set of critical values, of a family $\mathcal{H}$ for α and ε close to 0, and also preserves the projection onto the ε-axis.

Theorem 6.2. *At $\alpha = \varepsilon = 0$ the generating family $\mathcal{H}$ is (α,ε)-$\mathcal{A}$ equivalent to one of the following versal deformations in variables $(u,s) \in \mathbb{R}^2$ and parameters $(\alpha,\varepsilon) \in \mathbb{R}^2$:*

Case 1 $f_{30}g_{30} > 0$	**Beaks**	$\lambda_0 \neq \dfrac{b_3}{b_3 \pm a_3}, 0, 1$	$\mathcal{H} = (u, s^3 - su^2 + \varepsilon s)$
	Gull	$\lambda_0 = \dfrac{b_3}{b_3 \pm a_3}$	$\mathcal{H} =$ $(u, s^5 + s^4 + s^2u + \alpha s^3 + \varepsilon s)$
Case 2 $f_{30}g_{30} < 0$	**Lips**	$\lambda_0 \neq 0, 1$	$\mathcal{H} = (u, s^3 + su^2 + \varepsilon s)$

The classification follows on from the reduction given in §3 and the observation that the component of H that is linear in s and u does not depend on α (see equation (4) above). The complete proof depends upon a special version of the versality theorem which holds for (α,ε)-$\mathcal{A}$ versality; see the Appendix (Lemma A.1). $\square$

The local singularities for the two parallel inflexions case studied in this article fit into the general adjacency diagram for equidistants as follows:

$$\begin{tikzcd}
\text{Smooth} & \text{Cusp} \arrow[l] & & \text{Swallowtail} \arrow[ll] & \text{Butterfly} \arrow[l] \\
& & \text{Beaks} \arrow[ul] & & \\
& \text{Lips} \arrow[uu] & & \text{Gull} \arrow[uu] \arrow[ul] &
\end{tikzcd}$$

Note that the normal forms for lips and beaks in Proposition 6.2 do not contain α, that is they do not depend on the particular λ-equidistant where $\lambda = \lambda_0 + \alpha$ is close to a non-special value λ_0. Along the chord between a pair of parallel inflexions, which we call a *supercaustic chord*, every ε-family of equidistants, for a fixed non-special value of λ, undergoes either a beaks or lips bifurcation depending on whether the inflexions have the same or opposite sign. (For inflexions of the same sign, this happens when crossing the α-axis at constant $\alpha \neq 0$ in Figure 3; for inflexions of opposite sign the 'lips' of Figure 4 shrinks to a point and vanishes.) It is well-known that such bifurcations are not possible for Legendrian curves such as the λ-equidistants, that is where λ varies but ε remains fixed, since they alter the topology of the curves (see for example [1, p.60]).

In the case of inflexions of the same sign, where $f_{30}g_{30}$ is positive, the supercaustic chord contains two special values of λ_0 where the more degenerate gull singularity occurs with normal form as given in Proposition 6.2. Since (α, ε)-$\mathcal{A}$ equivalence preserves the fibration over ε, the normal form of the gull singularity preserves how the λ-equidistants bifurcate for fixed values of ε near zero ($\lambda = \lambda_0 + \alpha$ where λ_0 is a special value).

Using the normal form for gull in the table, we can calculate the critical set of $\mathcal{H}$ and hence the critical locus (set of critical values) which is the 'big equidistant'. From that a clock diagram can be drawn and this will correctly depict the bifurcations of the equidistants but only projection to the ε-axis can be relied on as representing the bifurcations for a fixed ε and α close to 0. Of course this clock diagram does not include any information about inflexions of the equidistants, such information not being preserved by diffeomorphisms. It is not difficult to calculate the swallowtail locus S and the self-tangency locus T in the (ε, α)-plane; these come to

$$\begin{aligned} S : \varepsilon &= -4s^3 - 15s^4, \alpha = -4s - 10s^2, \text{ so } \varepsilon = \tfrac{1}{16}\alpha^3 + \ldots; \\ T : \varepsilon &= s^4, \alpha = -2s^2, \text{ so } \varepsilon = \tfrac{1}{4}\alpha^2, \ \alpha \leq 0. \end{aligned}$$

A sketch of the resulting clock diagram is in Figure 6.

As previously mentioned, gull singularities also occur as projections of smooth surfaces in $\mathbb{R}^3$ to the plane (see for example [3, 5, 8]). In such projections, other singularities of the same codimension as the gull include butterfly and goose singularities. It is interesting to note that whilst butterfly and gull singularities both occur in one-parameter families of equidistants (only the latter in the context of our article), goose singularities are absent from the list. Goose singularities occur as the closure of the intersection of lips and beaks strata, and for equidistants these occur separately depending on the sign of $f_{30}g_{30}$. Since we assume our original curves have ordinary inflexions, so that f_{30} and g_{30} are nonzero, goose singularities do not occur.

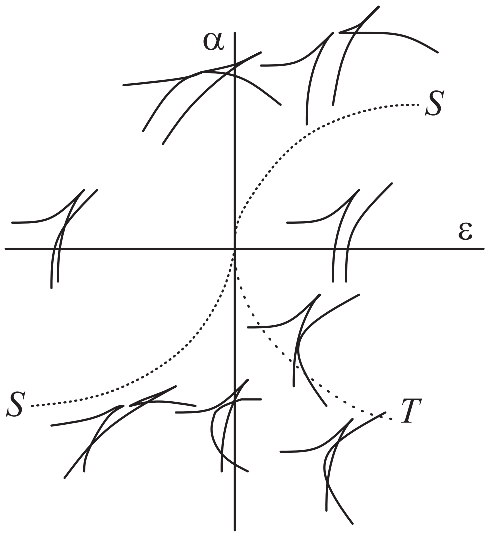

Fig. 6. The clock diagram for a gull singularity derived from the normal form in Theorem 6.2. Here S is the swallowtail set and T is the self-tangency set.

§Appendix A. Sketch of the proof of Theorem 6.2

Denote by Ω the space of map germs $F = (F_1, F_2) : \mathbb{R}^2 \times \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ in variables $x \in \mathbb{R}^2$, parameters α and ε, with the property that the linear terms in $x = (x_1, x_2)$ do not depend on α. That is, for each $i, j \in \{1, 2\}$ we have $\left.\dfrac{\partial^2 F_i}{\partial x_j \partial \alpha}\right|_{x=0} \equiv 0$.

Lemma A.1. *An infinitesimally (α, ε)-$\mathcal{A}$ versal germ $F \in \Omega$ is (α, ε)-$\mathcal{A}$ versal.*

Proof (This proof is modelled on [7]; see also [2, p.151].) Let F be an infinitesimal (α, ε)-$\mathcal{A}$ versal deformation of the germ $f : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and let G be a 1-parameter deformation of F (parameter β):

$$G(x, \alpha, \varepsilon, 0) \equiv F(x, \alpha, \varepsilon), \quad F(x, 0, 0) \equiv f(x)$$

$$G : (\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0).$$

We may consider G as an 3-parameter deformation of the germ of a map in x with parameters $\alpha, \varepsilon, \beta \in \mathbb{R}$. The result follows from the following lemma.

Lemma A.2. *The deformation G of f is (α,ε)-$\mathcal{A}$ equivalent to one induced from F.*

Proof Following [7, p.8–9] the key step in the argument is to solve the following "homological equation" for the unknown map germs Ξ_1, Ξ_2, L_1, L_2 and A.

$$\begin{aligned} & \frac{\partial G}{\partial \beta} + \frac{\partial G}{\partial \alpha}\Xi_1(\alpha,\varepsilon,\beta) + \frac{\partial G}{\partial \varepsilon}\Xi_2(\varepsilon,\beta) + \sum_{i=1}^{2}\frac{\partial G}{\partial x_i}L_i(x,\alpha,\varepsilon,\beta) \\ = \;& A(G(\alpha,\varepsilon,\beta,x),\alpha,\varepsilon,\beta), \end{aligned}$$

where for us it is important that Ξ_2 is a function of ε,β only, that is independent of α.

For any germ $s : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ there exists a decomposition signifying infinitesimal (α,ε)-$\mathcal{A}$ versality (here and below terms involving derivatives with respect to $x = (x_1, x_2)$ stand for the sum of two terms):

$$s(x) = \frac{\partial f}{\partial x}h(x) + k(f(x)) + \frac{\partial F}{\partial \alpha}\xi + \frac{\partial F}{\partial \varepsilon}\nu, \text{ where } \xi,\nu \in \mathbb{R}.$$

Consequently for every germ $S(x,\alpha,\varepsilon,\beta)$ there exists a decomposition:

$$\begin{aligned} S(x,\alpha,\varepsilon,\beta) \;=\; & \frac{\partial G}{\partial x}h(x) + K(G,\alpha,\varepsilon,\beta) + \frac{\partial G}{\partial \alpha}\xi + \frac{\partial G}{\partial \varepsilon}\nu \\ & +[\beta\sigma_0(x,\alpha,\beta,\varepsilon) + \alpha\sigma_1(x,\alpha) + \varepsilon\sigma_2(x,\alpha,\varepsilon)] \end{aligned}$$

Decompose σ_0, σ_1 and σ_2 using the same procedure:

$$\begin{aligned} S(x,\alpha,\varepsilon,\beta) \;=\; & \frac{\partial G}{\partial x}\left(h(x) + \beta h_0(x) + \alpha h_1(x) + \varepsilon h_2(x)\right) \\ & +\widetilde{K}(G(\alpha,\varepsilon,\beta,x),\alpha,\varepsilon,\beta) \\ & +\frac{\partial G}{\partial \alpha}\left(\xi + \beta\xi_0 + \alpha\xi_1 + \varepsilon\xi_2\right) \\ & +\frac{\partial G}{\partial \varepsilon}(\nu + \beta\nu_0 + \alpha\nu_1 + \varepsilon\nu_2) \\ & +[\beta^2\sigma_{00}(x,\alpha,\beta,\varepsilon) + \beta\alpha\sigma_{01}(x,\alpha,\varepsilon) + \alpha^2\sigma_{11}(x,\alpha,\varepsilon) \\ & \qquad +\beta\varepsilon\sigma_{02}(x,\varepsilon) + \alpha\varepsilon\sigma_{12}(x,\varepsilon) + \varepsilon^2\sigma_{22}(x,\varepsilon)] \end{aligned}$$

We have now obtained a better decomposition where the part in the square bracket is now of the second order in α,ε and β. The coefficients of the other terms will form the linear parts in α,ε,β of L_i, A, Ξ_1, Ξ_2 respectively.

Notice that if we consider mappings $S \in \Omega$ (i.e. with the property that no linear part in the variables contain α) then ν_1 will be zero and the coefficient of $\partial G/\partial\varepsilon$ will be a function in ε and β only. Continuing in this way we obtain the decomposition

$$\begin{aligned} S(x,\alpha,\varepsilon,\beta) &= \frac{\partial G}{\partial x}L(x,\alpha,\varepsilon,\beta) \\ &+ K(G(x,\alpha,\varepsilon,\beta),\alpha,\varepsilon,\beta) + \frac{\partial G}{\partial \alpha}\Xi_1(\alpha,\varepsilon,\beta) + \frac{\partial G}{\partial \varepsilon}\Xi_2(\varepsilon,\beta) \end{aligned} \tag{12}$$

at the level of a formal power series. Note that Ξ_2 does not depend on α. The preparation theorem (see [2]) shows that the decomposition (12) exists for convergent series and in the C^∞ case, where it is necessary to apply the preparation theorem to the $\mathcal{E}_{x,\alpha,\varepsilon,\beta}$ module $(\mathcal{E}_{x,\alpha,\varepsilon,\beta})^2/\{\frac{\partial G}{\partial x} + K(G(x,\alpha,\varepsilon,\beta),\alpha,\varepsilon,\beta)\}$, to the map $(y,\alpha,\varepsilon,\beta) \to (\alpha,\varepsilon,\beta)$ and to the generators $\frac{\partial G}{\partial \alpha}$ and $\frac{\partial G}{\partial \varepsilon}$. The decomposition (12) for $S = -\frac{\partial G}{\partial \beta}, K = -A$ provides the desired solution of the homological equation and the lemma is proved. $\square$

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Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK

E-mail address: `pjgiblin@liv.ac.uk, grahamreeve86@gmail.com`