Equidistants and their duals for families of plane curves

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Abstract

In this paper a Minkowski analogue of the Euclidean medial axis of a closed and smooth plane curve is introduced. Its generic local configurations are studied and the types of shocks that can occur on it are determined.

1 Introduction

The concept of the *medial axis* of Euclidean plane curves was first introduced by Blum in [1]. For a closed and smooth curve γ the medial axis is defined to be the locus of the centres of maximal circles that are tangent to γ in two or more points. Here a circle is said to be maximal if its radius equals the absolute minimum distance from its centre to γ: such a circle is either contained in the interior or the exterior of γ and cannot be expanded about its centre without crossing γ. Many applications of medial axes are given in Blum's original paper [1] and other applications relating to computer vision can be found in [10].

The *symmetry set* of the curve γ is the same as that of the medial axis except that the constraint that the circles must be maximal is dropped (see [4, 5]). For this reason the medial axis forms a subset of the symmetry set.

In [6] the generic shocks that can occur on the Euclidean medial axis are classified. Motivated by applications in fluid mechanics, Bogaevsky uses a different approach in [3] to obtain similar results to those in [6].

In this paper a Minkowski analogue of the Euclidean medial axis for a closed and smooth plane curve, called the Minkowski medial axis (MMA) is introduced. An analogue of the symmetry set for curves in the Minkowski plane, called the *Minkowski Symmetry Set (MSS)*, is introduced and studied in [11]. For a curve γ in the Minkowski plane, the MSS is defined as the locus of the centres of pseudo-circles that are tangent to the curve γ in two or more points.

Similarly to the Euclidean version, a point on the MSS is said to belong to the Minkowski medial axis if the radius r of the bi-tangent pseudo-circle equals the absolute maximum (if r is positive) or the absolute minimum (if r is negative) distance from its centre to γ. It is shown in Theorem 4.2 that this definition implies that the bi-tangent points lie on just one branch of the pseudo-circle. This fact leads on to a new generalised type of MMA, called the *1-branch MMA*. This is defined to be the centres of pseudo-circles that have a bi-tangency with just one branch of the pseudo-circles. It follows that the MMA forms a subset of the 1-branch MMA. The Minkowski symmetry set together with a radius function, like its Euclidean counterpart ([10]), can be used to reconstruct the original curve γ. Unlike with the Euclidean medial axis, neither the MMA nor the 1-branch MMA can be used to reconstruct non-convex curves. However, the 1-branch MMA together with a radius function can be used to reconstruct convex curves.

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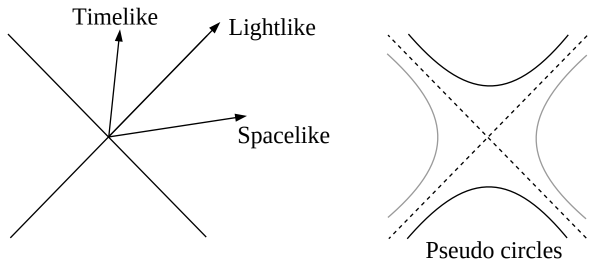

Figure 1: The three types of vectors (left) and pseudo-circles (right) in $\mathbb{R}^2_1$.

The *shock set* of γ is obtained by adding an arrow to the MMA indicating the direction of increasing radii of the relevant bi-tangent pseudo-circles. In this paper, the generic local configuration of the MMA and shocks of curves in the Minkowski plane are obtained.

2 Preliminaries

The *Minkowski plane* $(\mathbb{R}^2_1, \langle , \rangle)$ is the vector space $\mathbb{R}^2$ endowed with the pseudo-scalar product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = -u_0 v_0 + u_1 v_1$, for any $\boldsymbol{u} = (u_0, u_1)$ and $\boldsymbol{v} = (v_0, v_1)$. A vector $\boldsymbol{u} \in \mathbb{R}^2_1 \setminus \{0\}$ is called

spacelike if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$,
timelike if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$ or
lightlike if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$.

The norm of a vector is defined by $||\boldsymbol{u}|| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$.

The pseudo-circles in $\mathbb{R}^2_1$ with centre $c \in \mathbb{R}^2_1$ and radius r are defined as follows:

$$
\begin{aligned}
H^1(c, r) &= \{p \in \mathbb{R}^2_1 \,|\, \langle p - c, p - c \rangle = -r^2\} \text{ if } r < 0, \\
S^1(c, r) &= \{p \in \mathbb{R}^2_1 \,|\, \langle p - c, p - c \rangle = r^2\} \text{ if } r > 0, \\
LC^*(c) &= \{p \in \mathbb{R}^2_1 \setminus 0 \,|\, \langle p - c, p - c \rangle = 0\} \text{ if } r = 0.
\end{aligned}
$$

Observe that $LC^*(c)$, which is the set of points at zero distance from c, is the union of the two lines through c with tangent directions $(1, 1)$ and $(1, -1)$, with the point c removed. The pseudo-circle $H^1(c, -r)$ has two branches which can be parameterised by $c + (\pm r \cosh(t), r \sinh(t))$, $t \in \mathbb{R}$. The pseudo-circle $S^1(c, r)$ also has two branches which can be parameterised by $c + (r \sinh(t), \pm r \cosh(t))$, $t \in \mathbb{R}$. See Figure 1.

Let $\gamma : J \to \mathbb{R}^2_1$ be a smooth curve, where J is an open interval of $\mathbb{R}$ or the unit circle S^1 if the curve is closed. The curve γ is spacelike if $\gamma'(t)$ is a spacelike vector for all $t \in J$ and is timelike if $\gamma'(t)$ is a timelike vector for all $t \in J$. A point $\gamma(t)$ is called a lightlike point if $\gamma'(t)$ is a lightlike vector.

If γ is spacelike or timelike, then it can be reparameterised by arc-length and its curvature is well defined at each point (see, for example, [11]). One can also have the notion of vertices (points where the derivative of the curvature vanishes). The curvature of γ is not defined at lightlike points (so we have no notion of vertices at such points). An inflection can be defined in terms of the contact of the curve with lines, so the curve can have an inflection at a lightlike point. For a generic curve, the lightlike points are not inflection points.

It is shown in ([11], Proposition 2.1) that the set of lightlike points of a closed curve γ is the union of at least four disjoint non-empty and closed subsets of γ (Figure 2). The complement of these sets are disjoint connected spacelike or timelike pieces of the curve γ.

The Minkowski symmetry set is defined in [11] as follows.

The Minkowski Symmetry Set (MSS) of a curve γ in the Minkowski plane is the closure of the locus of centres of bi-tangent pseudo-circles to the curve.

In the Minkowski plane $\mathbb{R}^2_1$, the fact that vectors can have negative length has to be considered. For this reason, a pseudo-circle is said to be *maximal* if its radius equals the absolute minimum *modulus* distance

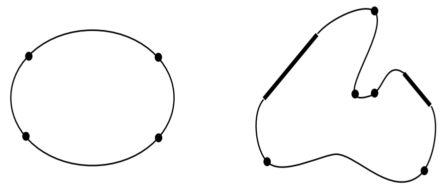

Figure 2: Lightlike points in thick on smooth closed curves in $\mathbb{R}^2_1$.

from its centre to γ. The radius of such a circle cannot be increased if it is of type $S^1(p,r)$ or decreased if it is type $H^1(p,r)$ without it crossing γ.

The *Minkowski medial axis* (MMA) of a curve γ in $\mathbb{R}^2_1$ is the subset of the Minkowski symmetry set formed by the centres of bi-tangent pseudo-circles which are maximal (see Figure 3).

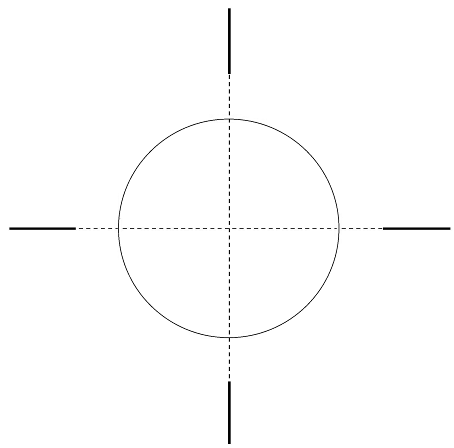

Figure 3: A circle and its MSS (the two transverse line segments). The MMA is the subset of the MSS represented by a thick line.

In the Euclidean plane, for a closed curve γ, maximality implies that the bi-tangent circles are either entirely inside or entirely outside the curve ([7]). In the Minkowski plane however, since pseudo-circles are not compact, maximality ensures that the centres are entirely outside the curve, see Lemma 4.1.

The family of distance-squared functions $f : J \times \mathbb{R}^2_1 \to \mathbb{R}$ on γ is given by

$$f(t,c) = \langle \gamma(t) - c, \gamma(t) - c \rangle,$$

and the *extended family of distance-squared functions* $\tilde{f} : J \times \mathbb{R}^2_1 \times \mathbb{R} \to \mathbb{R}$ is given by

$$\tilde{f}(t,c,r) = \langle \gamma(t) - c, \gamma(t) - c \rangle - r^2.$$

(To simplify notation the extended family of distance-squared functions will also be denoted by f.)

Denote by $f_c : J \to \mathbb{R}$ the function given by $f_c(t) = f(t,c)$. The function f_c is said to have an A_k-singularity at t_0 if $f_c'(t_0) = f_c''(t_0) = \ldots = f_c^{(k)}(t_0) = 0$ and $f_c^{(k+1)}(t_0) \neq 0$. This is equivalent to the existence of a local re-parametrisation h of γ at t_0 such that $(f \circ h)(t) = \pm(t - t_0)^{k+1}$.

If f_c has a singularity at t_1 of type A_k and at t_2 of type A_l, then f is said to have a multi-local singularity of type $A_k A_l$.

Geometrically, f_c has an A_k-singularity if and only if the curve γ has contact of order $k+1$ at $\gamma(t_0)$ with the pseudo-circle $C(c,r)$ of centre c and radius r, with $|r| = ||\gamma(t_0) - c||$. The distance squared function f_c

has A_kA_l-singularity if and only if the pseudo circle $C(c,r)$ is tangent to γ at two distinct points and has order of contact $k+1$ at one of them and $l+1$ at the other.

It follows from Thom's transversality theorem (see for example [2, 8]) that for an open and dense set of immersions $\gamma : S^1 \to \mathbb{R}^2_1$ the function f_c has only local singularities of type A_1, A_2, A_3 and multi-local singularities of type A_1^2, A_1A_2, A_1^3.

The MSS is the multi-local component of the bifurcation set of the family f, that is,

$$MSS = \{c \in \mathbb{R}^2_1 \,|\, \exists t_1, t_2 \text{ such that } t_1 \neq t_2,\ f_c(t_1) = f_c(t_2),\ f_c'(t_1) = f_c'(t_2) = 0\}.$$

It follows from Theorem 3.2 in [9] that the family f is always a versal unfolding of its generic singularities, so the MSS is locally diffeomorphic to the bifurcation set of the models of such singularities (Corollary 3.3 in [9]). Thus the configuration of the MSS at the generic multi-local singularities of f_c are as in Figure 4.

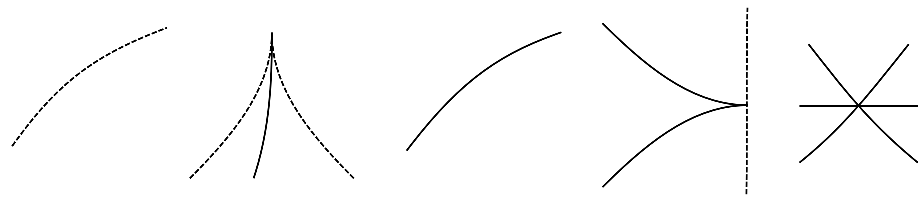

Figure 4: Generic local models of the MSS in continuous line (the dashed curve is the caustic of the curve). Only the A_1^2, A_1^3 and A_3 singularities occur generically on the MMA.

For a point on the MSS to also belong to the MMA the relevant bi-tangent pseudo-circle must be maximal (Definition 2). This means that the MMA forms a subset of the MSS and in particular this condition ensures that only A_1^2, A_3 and A_1^3 can belong to MMA. This is because for the generic singularity types A_2 and A_1A_2 the pseudo-circle locally crosses the curve and therefore it cannot be maximal.

In [11] it was shown that the MSS is a regular curve at c_0 if and only if the bi-tangent pseudo-circle to γ at $\gamma(t_1)$ and $\gamma(t_2)$ is not osculating at $\gamma(t_1)$ or at $\gamma(t_2)$. If this is the case, the tangent line to the MSS at p is the perpendicular bisector to the chord joining $\gamma(t_1)$ and $\gamma(t_2)$. (This is also true in the Euclidean case, see [5].)

3 Local reconstruction of the curve from the MSS

Suppose we are given the MSS (or MMA) of a spacelike or timelike smooth curve γ. Then the MSS is either a spacelike or a timelike curve [11]. If the MSS is not singular, it can be parameterised by arc length $c(s) = (x(s), y(s))$ and denote by $r(s)$ the radius of the bi-tangent circle to γ centred at $c(s)$. Then it is possible to reconstruct local parametrisations γ_1 and γ_2 of the two corresponding arcs of γ as an envelope of the bi-tangent circles $C(c(s), r(s))$.

If the curve $c(s)$ is timelike and the bi-tangent pseudo-circle is of type $S^1(p,r)$ or if the curve $c(s)$ is spacelike and the bi-tangent pseudo-circle is of type $H^1(p,r)$, then the points of tangency are given by

$$\gamma_i = c + r\left(\frac{\partial r}{\partial s}\right)T + (-1)^i\left(r\sqrt{\left(\frac{\partial r}{\partial s}\right)^2 + 1}\right)N,\ i = 1, 2, \tag{1}$$

where T and N are the unit tangent and unit Minkowski normal to the MSS and r is the radius of the bi-tangent pseudo-circle, all evaluated at s.

If the curve $c(s)$ is spacelike and the bi-tangent pseudo-circle is of type $S^1(p,r)$ or if the curve $c(s)$ is timelike and the bi-tangent pseudo-circle is of type $H^1(p,r)$, then the points of tangency are given by

$$\gamma_i = c - r\left(\frac{\partial r}{\partial s}\right)T + (-1)^i\left(r\sqrt{\left(\frac{\partial r}{\partial s}\right)^2 - 1}\right)N,\ i = 1, 2. \tag{2}$$

Proof:
Suppose that c is spacelike and the pseudo-circles are of type $H^1(p,r)$. Then the pseudo-circle of radius $r(s)$ centred at $c(s)$ is the set of points $w \in \mathbb{R}^2_1$ such that

$$F(s,w) = \langle c(s) - w, c(s) - w \rangle + r(s)^2 = 0.$$

The envelope of this family of pseudo-circles is given by

$$D(F) = \{w \in \mathbb{R}^2 : \exists s \in \mathbb{R} \text{ such that } F(s,w) = \frac{\partial F}{\partial s}(s,w) = 0\}.$$

Differentiating F with respect to s and dropping the arguments yields

$$\frac{\partial F}{\partial s} = 2\langle (c - w, T\rangle + 2r\frac{\partial r}{\partial s}.$$

Since γ is spacelike, $\langle T,T\rangle = 1$ and $\langle N,N\rangle = -1$. Writing $c - w = \lambda T + \mu N$, with $\lambda, \mu \in \mathbb{R}$, and substituting into $\partial F/\partial s = 0$ yields $\lambda + r\partial r/\partial s = 0$ so that $\lambda = -r(\partial r/\partial s)$. Substituting into $F = 0$ yields $\lambda^2 - \mu^2 + r^2 = 0$ so that $\mu = \pm r\sqrt{(\partial r/\partial s)^2 + 1}$. It follows that the locii of the envelope points are as given (1). The same method can be applied to find the formula for the envelope in the remaining cases. □

In the Euclidean case the rate of change of the radius function is restricted to be less than or equal to 1 in order for the envelope to be real, see [7]. In the Minkowski setting this restriction only applies when the pseudo-circles and the curve c are of opposite type (spacelike/timelike), otherwise any function r gives a real envelope.

Theorem 4.1 of [9] states that for any point p of a closed smooth curve γ there exists another point $q \in \gamma$ and a pseudo-circle that is tangent to γ at both p and q. From this, and the fact that it is possible to reconstruct the curve locally, it follows that it is possible to reconstruct any smooth closed curve from its Minkowski symmetry set.

4 The Minkowski medial axis

The *Minkowski hull* $MH(\gamma)$ of a smooth closed curve γ in the Minkowski plane is the region of the plane such that for any point $p \in MH(\gamma)$ there exists a point $q \in \gamma$ such that the Minkowski distance between p and q is zero.

The complement of the Minkowski hull of a closed smooth curve γ in the Minkowski plane consists of four disjoint open regions.

Proof: Consider a smooth closed curve $\gamma : S^1 \to \mathbb{R}^2_1$ and use coordinates in $\mathbb{R}^2_1$ such that the axes are parallel to the lightlike directions. We write $\gamma(t) = (x(t), y(t))$. Since the functions $x(t)$ and $y(t)$ are bounded, they must both attain an absolute maximum and an absolute minimum. Since the curve is smooth, this gives exactly four extremal sets (two of $x(t)$ and two of $y(t)$) on the curve. The curve γ is now contained in the compact region determined by the four tangent lines to points in these sets. The tangent lines divide the plane into the Minkowski hull and four disjoint regions of its complement, see Figure 5. □

Lemma 4.1 *The points on the MMA of a closed plane curve which are the centres of bi-tangent circles of type $H^1(p,r)$ or $S^1(p,r)$ lie in the complement of the Minkowski hull of the curve.*

Proof: From the definition, a point p belonging to the Minkowski hull has Minkowski distance zero to some point on the curve. It follows that pseudo-circles of types $H^1(p,r)$ and $S^1(p,r)$ cannot be maximal because the light cone $LC^*(p)$ would intersect the curve. Therefore, pseudo-circles of types $H^1(p,r)$ and $S^1(p,r)$ corresponding to the MMA lie in the complement of the Minkowski hull. □

The MMA of a closed smooth convex plane curve lies in the closure of the complement of the Minkowski hull.

Proof: Closed smooth convex plane curves have exactly four closed lightlike regions (see [11]). The four centres of the pseudo-circles that are bi-tangent to these four regions are on the boundary of the Minkowski

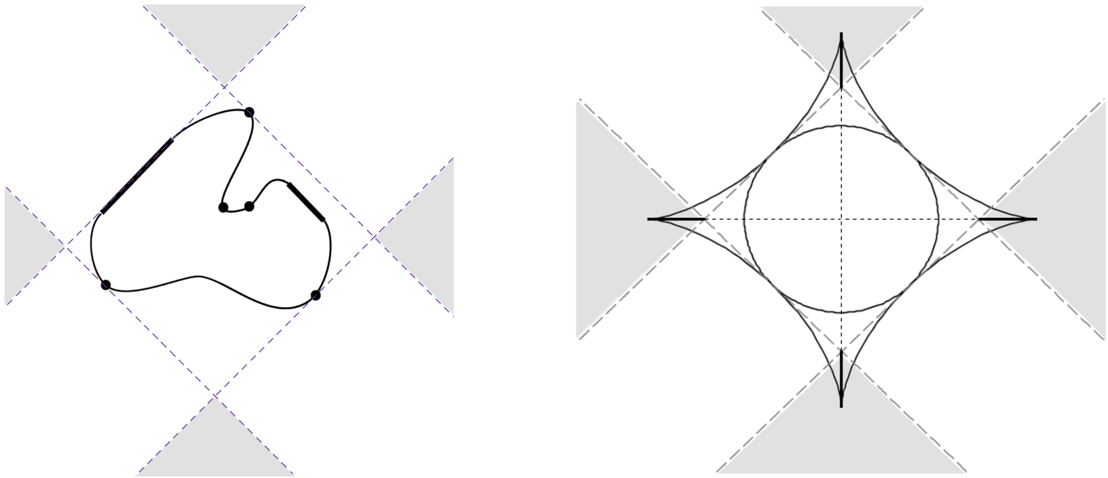

Figure 5: Left: the Minkowski hull of a closed curve in the Minkowski plane and its complement (shaded in grey). Right: a circle, its Minkowski caustic (continuous curve with four cusps), Minkowski hull (shaded in grey), MSS and MMA (in thick).

hull. This, together with Lemma 4.1, prove that the centres of all three types of maximal bi-tangent pseudo-circles lie in the complement of the Minkowski hull or on its boundary. Therefore, the MMA of a closed smooth convex plane curve lies in the closure of the complement of the Minkowski hull. See Figure 5, right. □

(of Lemma 4.1.) The MMA of a closed smooth (not necessarily convex) plane curve lies in the complement of the Minkowski hull of the curve except for the centres of bi-tangent pseudo-circles of type $LC(p)$ which lie on the boundary of the complement of the Minkowski hull (note that these centres form subsets of lightlike lines).

It is worth observing the difference between the medial axis of a convex curve in the Euclidean plane and of the MMA of a convex curve in the Minkowski plane. The former lies strictly inside the curve while the latter lies strictly outside the curve.

A pseudo-circle of type $H^1(p,r)$ or $S^1(p,r)$ is said to be *1-branch bi-tangent* to a curve γ if one of its branches is tangent to γ in at least two distinct points.

Since the MMA lies in the closure of the complement of the Minkowski hull we have the following result.

Theorem 4.2 *The pseudo-circles of types $H^1(p,r)$ and $S^1(p,r)$ corresponding to the MMA of a closed plane curve are all 1-branch bi-tangent.*

Proof: Lemma 4.1 states that the centres of the bi-tangent pseudo-circles of type $H^1(p,r)$ and $S^1(p,r)$ lie in the complement of the Minkowski hull of the curve. Since the bi-tangent pseudo-circle necessarily has one of its branches completely contained in the complement of the Minkowski hull it follows that the tangencies must occur on only one of the pseudo-circle's two branches. □

Note that the converse of Theorem 4.2 is true for convex curves but is not true for non-convex curves. That is, not all 1-branch bi-tangent pseudo-circles are maximal for non-convex curves. See for example Figure 7 (right).

One of the useful properties of the Euclidean medial axis is that it can be used to reconstruct the original curve. It is shown in Proposition 4.5 of [9] that for the piece of curve $\alpha(t) = (t, t^3)$, $-\frac{\sqrt{3}}{3} < t < \frac{\sqrt{3}}{3}$, there are no 1-branch bi-tangent pseudo-circles. The curve $\alpha(t)$ can be extended smoothly to obtain a closed curve γ. It is possible to construct γ so that there do not exist pseudo-circles which are 1-branch tangent to a point of $\alpha(t)$ and are also tangent to some other point of $\gamma(t)$. Consider for example the limaçon whose radius r is given by $r = \frac{3}{2} + \cos(\theta)$ where $-\pi < \theta < \pi$. Splitting the limaçon into timelike and spacelike components, only pairs of points from the regions $\arctan(\frac{\sqrt{35}}{17}) - \pi < \theta < -\arctan(\frac{\sqrt{35}}{17}) + \pi$ have corresponding Minkowski medial axis points, see Figure 6. Therefore, the MMA together with a radius function (MMA transform), unlike its Euclidean counterpart, is not a complete shape describer.

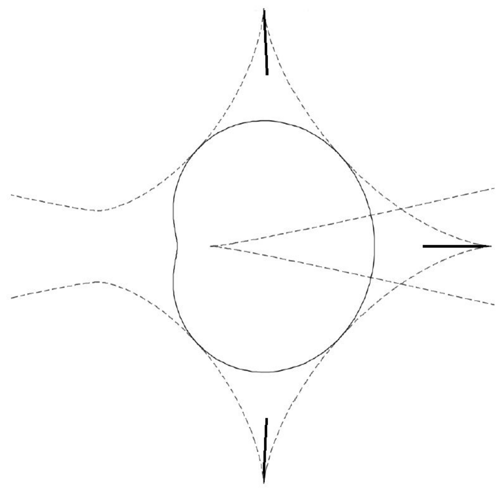

Figure 6: The limaçon given by $r = \frac{3}{2} + \cos(\theta)$, $-\pi < \theta < \pi$, its Minkowski caustic (dashed) and its MMA (thick curves).

The fact that the MMA consists of centres of only 1-branch bi-tangent pseudo-circles motivates the following new type of medial axis, which shall be called the *1-branch Minkowski medial axis*.

The 1-branch Minkowski medial axis is the locus of the centres of pseudo-circles that are tangent to the curve γ in two or more points such that the tangencies occur on just one of the branches of the pseudo-circle.

Since the MMA is made up of *only* 1-branch bi-tangent pseudo-circle centres, it is a subset of the 1-branch medial axis.

The two sets are not equal because not all 1-branch bi-tangent pseudo-circles are maximal, see for example Figure 7, left. The 1-branch Minkowski medial axis does not lie in the complement of the Minkowski hull for non-convex curves; see for example Figure 7, right.

Theorem 4.5 of [9] states that for any point p on a spacelike or timelike curve γ without inflections there exists another point q on γ and a pseudo-circle that is tangent to γ at both p and q with both points being on a single branch of the pseudo-circle. From this it follows that any closed convex curve can be reconstructed from its 1-branch medial axis. The four lightlike components are either isolated points or lightlike line segments (in the generic case only isolated points are possible). To complete γ, these components can be added by taking the closure of the curve if they are just isolated points, or by joining up the remaining components with lightlike lines.

5 Shocks on the Minkowski medial axis

At each point on the medial axis of a curve γ in the Euclidean plane there is an associated radius function r corresponding to the radius of the bi-tangent circle. The direction of the increasing radius function on the medial axis, that is the direction for which $\partial r/\partial s(s) > 0$, can be indicated by an arrow and this gives the shock set (see for example [1, 6]).

The shock set is a dynamic view of the medial axis. If there is a propagation of waves from the curve γ, then this leads to the formation of singularities (the medial axis). The shock set gives the direction along which this formation of singularities propagates.

In [6] the local generic forms of shocks that can occur on the medial axis in the Euclidean plane are classified. It is shown that some types of shocks cannot occur generically on the Euclidean medial axis. For example, it is proven that the only form of shock that can occur at an A_3-singularity of a given distance squared function on γ is that with outward velocity, see Figure 8.

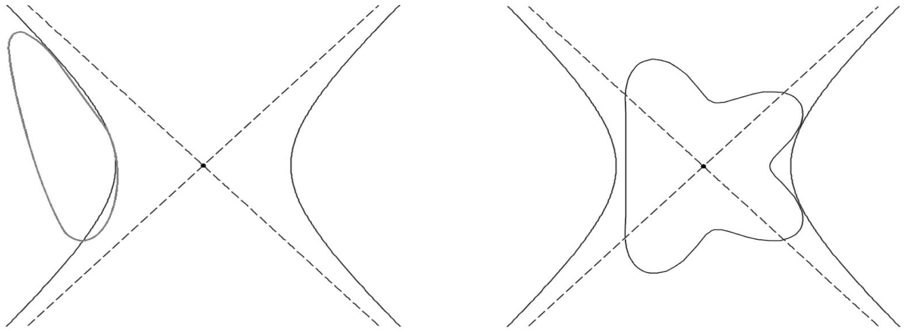

Figure 7: Left: an illustration of a 1-branch bi-tangent pseudo circle which is not maximal (A_1A_2-singularity). Right: a 1-branch bi-tangent pseudo-circle whose centre lies in the Minkowski hull of the (non-convex) curve.

We define the analogue of shocks in the Minkowski plane. This is the Minkowski medial axis together with an arrow in the direction of increasing radius of the corresponding bi-tangent pseudo-circle.

In this section it is shown that for the MMA both types of shock can occur (outwards and inwards) at an A_3-singularity of a given distance squared function on γ, depending on whether the MMA is spacelike or timelike. It will also be shown that the generic shocks that can occur at an A_1^3-singularity of a given distance squared function on γ are different to those on the Euclidean medial axis.

In what follows, the singularities refer to those of a given distance squared function.

5.1 Shocks at an A_3-singularity

The A_3-singularity occurs at a vertex of the curve. These occur where the two A_1-contact points for nearby A_1^2-bi-tangent pseudo-circles come into coincidence. The A_1-points must therefore lie on the same branch of the pseudo-circle. This means that given a MMA near an A_3 point and its associated radius function, formula (2) can be used to find the corresponding points on γ.

Theorem 5.1 *If the curve γ is timelike/spacelike at an ordinary vertex, then the shock on the MMA is of outwards/inwards type respectively.*

Proof: Consider a neighbourhood of an A_3 point on a spacelike MMA (which necessarily corresponds to a timelike piece of γ, see Theorem 5.2 in [9]). Orient the MMA so that its tangent line points towards the branch of the bi-tangent pseudo-circle that contains the tangent points. In the coordinate system with origin at the A_3 point on MMA and with basis T and N, the envelope points of the bi-tangent pseudo-circles have positive T coordinate. It follows from formula (2) that $-r\partial r/\partial s > 0$. As $r > 0$, $\partial r/\partial s < 0$ so the radius function r must decrease in the direction of the A_3 point. Therefore, for a timelike γ near a vertex, the shocks are of outwards type; see Figure 8.

If the MMA is timelike (the curve γ must be spacelike, Theorem 5.2 in [9]). Following the same arguments as above and using formula (2) at an A_3-point, yields $-r\partial r/\partial s > 0$. Here $r < 0$ so it follows $\partial r/\partial s > 0$. This implies that the shocks are of inwards type; see Figure 8. $\square$

Theorem 5.1 also holds when the MMA is replaced by the 1-branch MMA.

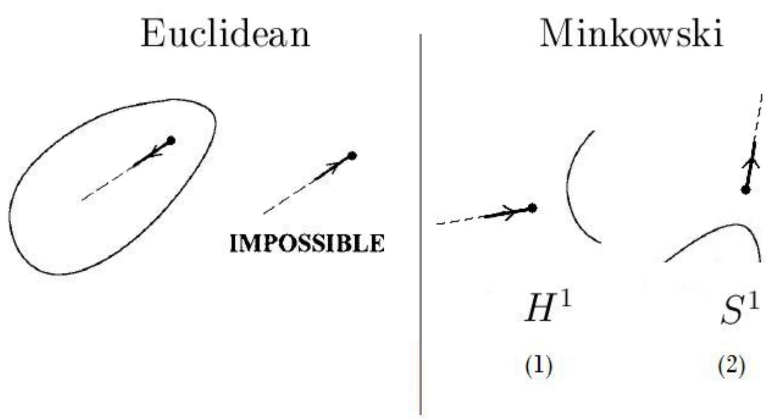

Figure 8: Comparing A_3 shocks that can occur on the Euclidean and Minkowski medial axes. In the Minkowski plane, both cases can occur and are distinguished by the type of the bi-tangent pseudo-circle.

5.2 Shocks at an A_1^3-singularity

As with the shocks at A_3-singularity, the shocks at A_1^3-singularity turn out to be different from those of the Euclidean medial axis. The type of shocks that can occur depends on whether the bi-tangent pseudo-circle is of type $H^1(p,r)$ or $S^1(p,r)$.

For closed curves, the MMA only consists of the centres of pseudo-circles whose tangencies occur on only one of its branches (Theorem 4.2). Theorem 5.3 gives a classification of shocks that can occur for closed curves. For two disjoint pieces of curves, it is possible that the pseudo-circle centred on the medial axis can be tangent to each piece of curve (so the centre is not on the 1-branch MMA). For completeness, Theorem 5.4 gives the classification of shocks at an A_1^3-singularity when the relevant pseudo-circle has at least one tangency on each of its branches.

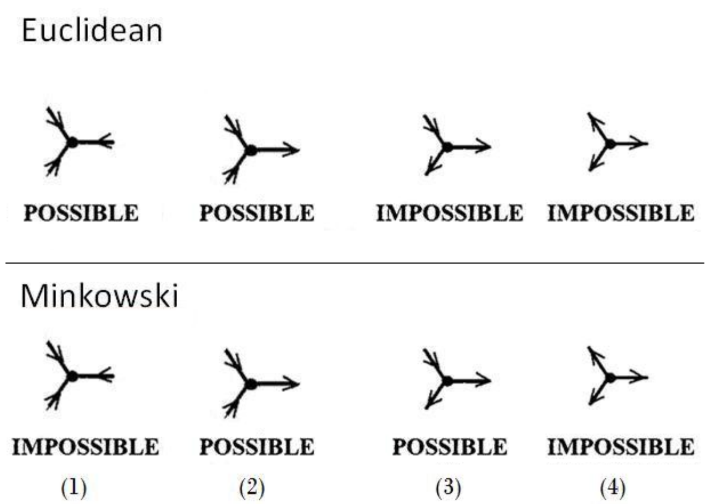

Figure 9: A_1^3 shocks that can occur on the Euclidean and Minkowski medial axes for closed curves. Case (2) occurs when the pseudo-circle is of type S^1. Case (3) occurs when the pseudo-circle is of type H^1.

Suppose that γ is a closed plane curve and the three tri-tangent points, say q_1, q_2 and q_3 on γ to a pseudo-circle all lie on one branch of the pseudo-circle. Each branch of the MMA corresponds to the centres of bi-tangent pseudo-circles whose points of tangency are near two of the points $q_i, i = 1, 2, 3$. For each branch, the two corresponding tangency points are called the characteristic points and are denoted X^- and X^+. If the branch corresponds to points near q_i and q_j then the point q_k with $k \neq i, j$ is denoted by P.

Lemma 5.2 *If the arc that contains X^- and X^+ does not contain the point P, then the medial axis goes in the direction of entering the arc. If the arc contains the point P then the medial axis goes away from the arc.*

Proof: Consider the case of a tri-tangent pseudo-circle of type $S^1(p, r)$, the proof is similar for a tri-tangent pseudo-circle of type $H^1(p, r)$ and is omitted.

Consider the function $f(s) = \langle c(s) - P, c(s) - P\rangle - (r(s))^2$, where s is the arc-length parameter of the MSS branch $c(s)$ and $r(s)$ is the radius of the bi-tangent pseudo-circle. Let s_0 correspond to the A_1^3 point. Note that $f(s_0) = 0$. We have tangency of type A_1, so $f'(s_0) \neq 0$. If $f'(s_0) < 0$, then $f(s) < f(s_0)$ for small $s > s_0$. For such s, $c(s)$ cannot be on the MMA since the point P will have come 'inside' the pseudo-circle centre $c(s)$ radius $r(s)$. Here, 'inside' means that its absolute distance from the centre is less than $|r|$.

Now $f'(s) = 2\langle c(s) - P, T(s)\rangle - 2r(s)r'(s)$, so that $f'(s_0) < 0$ is equivalent to $\langle c(s_0) - P, T(s_0)\rangle < r(s_0)r'(s_0)$. We have $r(s_0)r'(s_0) = \langle c(s_0) - X^\pm, T(s_0)\rangle$. This in turn implies $\langle X^\pm - P, T\rangle < 0$.

Parameterise the tri-tangent pseudo-circle by $g(t) = (r_0 \sinh(t), r_0 \cosh(t))$. Suppose that $X^+ = g(\theta)$, $X^- = g(\varphi)$ and $P = g(\rho)$ for some θ, φ and ρ.

The tangent to the branch of the MSS corresponding to the characteristic points X^+ and X^- has direction $T = (\sinh(\frac{\theta+\varphi}{2}), \cosh(\frac{\theta+\varphi}{2}))$ and the vector $X^+ - p$ has direction $(\cosh(\frac{\theta+\rho}{2}), \sinh(\frac{\theta+\rho}{2}))$ if $\theta > \rho$ and $(-\cosh(\frac{\theta+\rho}{2}), -\sinh(\frac{\theta+\rho}{2}))$ if $\theta < \rho$. Assume that $\theta > \rho$. The Minkowski product $\langle X^\pm - P, T\rangle$, up to a positive factor, is given by

$$-\sinh\left(\frac{\theta+\varphi}{2}\right)\cosh\left(\frac{\theta+\rho}{2}\right) + \sinh\left(\frac{\theta+\rho}{2}\right)\cosh\left(\frac{\theta+\varphi}{2}\right) = 2(e^\rho - e^\varphi)e^{-(\theta+\varphi+\rho)}$$

which is negative if and only if $\rho < \varphi$.

Similarly, assuming $\theta < \rho$ gives that $\langle X^\pm - P, T\rangle$ is negative if and only $\varphi < \rho$.

Thus the condition $\langle X^\pm - P, T\rangle < 0$ is equivalent to either $\rho < \varphi, \theta$ or $\rho > \varphi, \theta$ which is equivalent to the statement in Lemma 5.2. □

Theorem 5.3 *For a smooth closed curve, the Minkowski medial axis at an A_1^3-singularity has shock type* (2) *in* Figure 9 *if the tri-tangent pseudo-circle is of type $S^1(p, r)$. It is of type* (3) *in* Figure 9 *if the tri-tangent pseudo-circle is of type $H^1(p, r)$.*

Proof: Consider tangent pseudo-circles of type $S^1(p, r)$, the other case follows similarly. For three points on the same branch of the pseudo-circle and for ρ, φ and θ as in the proof of Lemma 5.2, the conditions $\rho < \varphi, \theta$ or $\rho > \varphi, \theta$ must be true for two of the three medial axis branches. The arrows, indicating directions of increasing radius, can now be added to the medial axes. Formula (2) implies that the radius must be increasing in the direction of the branch of the pseudo-circle that contains the three tri-tangent points. □

Theorem 5.3 gives a complete classification of shocks for closed curves at an A_1^3-singularity. It also holds when the MMA is replaced by the 1-branch MMA. In this case the condition that the curve be closed can also be dropped.

Blum viewed the (Euclidean) medial axis as a quench point for grass-fire flow initiated from the boundary of the shape [1]. He considered the medial axes of curve segments as well as closed curves. In this spirit, and for the purpose of applications, we now consider the MMA and classify its shocks for when the tri-tangency occurs on both branches of the pseudo-circle.

Theorem 5.4 *The shocks that can occur at an A_1^3-singularity when the tangent points occur on both branches of a pseudo-circle of type $S^1(p, r)$ are as shown in* Figure 10. *For pseudo-circles of type $H^1(p, r)$ the shocks are as in* Figure 10 *with the directions of the arrows reversed.*

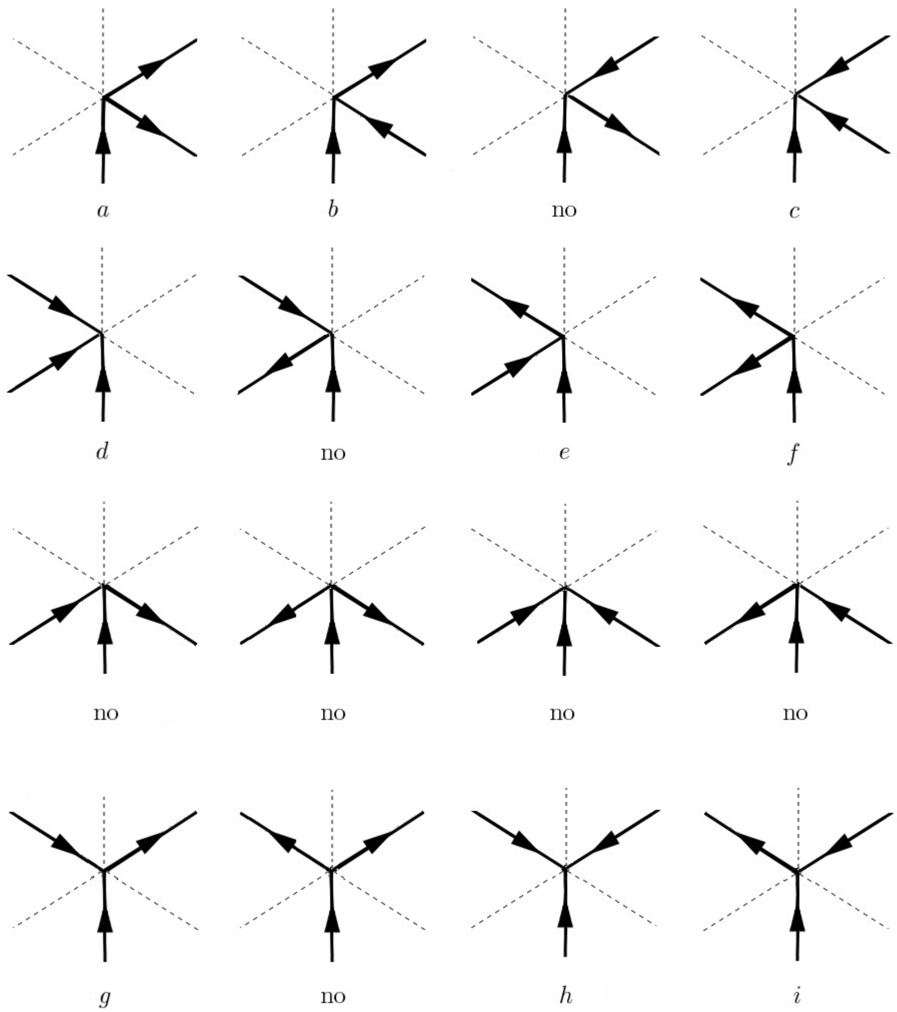

Figure 10: The types of shocks that can occur for timelike A_1^3 points.

Proof: Consider pseudo circles of type $S^1(p,r)$; the proof is similar for the pseudo-circles of type $H^1(p,r)$ and is omitted. Take the centre of the tri-tangent pseudo-circle to be the origin. The pseudo-circle has two branches, parameterise them by $g_1(t) = (r_0 \sinh(t), r_0 \cosh(t))$ and $g_2(t) = (r_0 \sinh(t), -r_0 \cosh(t))$ and assume that the branch containing two tangent points to be the branch in the lower half of the plane. Denote by $X_\theta = g_1(\theta)$ the tangent point on the upper branch and by $X_\varphi = g_2(\varphi)$ and $X_\rho = g_2(\rho)$ the two tangent points on the lower branch.

Denote by $T_{\theta,\varphi}$ the tangent line to the MMA branch that corresponds to the characteristic points θ and φ and similarly for $T_{\theta,\rho}$ and $T_{\rho,\varphi}$. Consider first the branch of the MMA with tangent line $T_{\rho,\varphi}$. This tangent line has direction

$$T_{\rho,\varphi} = (\sinh(\frac{\rho+\varphi}{2}), \cosh(\frac{\rho+\varphi}{2}))$$

and the vector joining the point X_ρ to X_θ is given by

$$X_\varphi - X_\theta = r_0(\sinh(\varphi) - \sinh(\theta), \cosh(\varphi) + \cosh(\theta)).$$

Taking their Minkowski product yields

$$\langle X_\varphi - X_\theta, T_{\rho,\varphi}\rangle = \langle X_\rho - X_\theta, T_{\rho,\varphi}\rangle = r_0 \cosh\left(\frac{\varphi-\rho}{2}\right) + r_0 \cosh\left(\frac{\varphi+\rho}{2} + \theta\right)$$

which is always positive. Therefore, the corresponding branch of the MMA starts at the A_1^3 point and goes in the direction of the branch containing the characteristic points. Formula (2) implies that the shock travels along this MMA branch in the direction towards the centre.

Consider now the two branches with tangents $T_{\theta,\rho}$ and $T_{\theta,\varphi}$. For each branch of the MMA there are four possibilities: The MMA can be to the left or to the right of the centre (since the branches are both timelike), and in both cases the direction of the shock can be either towards or away from the centre.

The tangent line $T_{\theta,\varphi}$ to the medial axis at the A_1^3, up to a nonzero factor, can be written

$$T_{\theta,\varphi} = \left(\frac{\sinh(\theta)+\sinh(\varphi)}{2}, \frac{\cosh(\theta)-\cosh(\varphi)}{2}\right)$$

and the vector joining the point X_ρ to X_θ, is given by $X_\theta - X_\rho = r_0(\sinh(\theta) - \sinh(\rho), \cosh(\theta) + \cosh(\rho))$. Taking their Minkowski product yields

$$\langle X_\theta - X_\rho, T_{\theta,\varphi}\rangle = \frac{r_0}{2}(1 - \cosh(\rho-\theta) + \cosh(\rho+\varphi) - \cosh(\theta+\varphi)) \quad (3)$$

which is positive when $-\varphi < \theta < \rho$ or $\rho < \theta < -\varphi$ and negative otherwise. This can be shown by considering all the possible orderings of $\theta, -\varphi$ and ρ. For example, suppose that $\theta < -\phi < \rho$ so that $\theta = -\phi - \delta$ for some positive δ. Substituting in (3) yields

$$\langle X_\theta - X_\rho, T_{\theta,\varphi}\rangle = \frac{r_0}{2}(1 - \cosh(\rho+\phi+\delta) + \cosh(\rho+\varphi) - \cosh(-\delta)).$$

The above expression is always negative because $\cosh(\rho+\varphi+\delta) > \cosh(\rho+\varphi)$ as both $\rho+\varphi$ and δ are positive, and $1 - \cosh(-\delta)$ is also negative.

Recall that when $\langle X_\theta - X_\rho, T_{\theta,\varphi}\rangle$ is positive, small positive $s > s_0$ belongs to the MMA, and when it is negative, it is small $s < s_0$ that belongs to the MMA. Now, the direction of increasing radius can be added to the MMA. It follows directly from formula (1) that if $\theta + \varphi > 0$ the radius function on the branch corresponding to $T_{\theta,\varphi}$ increases from left to right, whereas if $\theta + \varphi < 0$ the radius increases from right to left, and similarly for $T_{\theta,\rho}$.

Considering the two branches together now, it must be determined which branch goes 'over' the other. Comparing the gradients of the two tangent lines it can be seen that $T_{\theta,\varphi}$ is steeper than $T_{\theta,\rho}$ if and only if

$$\theta < \rho < \varphi,\ \rho < \varphi < \theta \text{ or } \varphi < \theta < \rho.$$

Otherwise, $T_{\theta,\rho}$ is the steeper of the two.

Considering these conditions, along with the above conditions for which side the branches lie on, gives the complete list of possible shocks that can occur (see Table 1 and Figure 10). □

Table 1: Conditions for shocks in Figure 10 to occur.

a)	$-\rho < -\varphi < \theta < \varphi < \rho$ $-\varphi < -\rho < \theta < \rho < \varphi$	b)	$-\rho < \theta < \varphi < -\varphi < \rho$ $-\rho < \theta < -\varphi < \varphi < \rho$ $-\varphi < \theta < -\rho < \rho < \varphi$ $-\varphi < \theta < -\rho < \rho < \varphi$
c)	$\theta < -\varphi, \varphi, -\rho, \rho$	d)	$-\varphi, \varphi, -\rho, \rho < \theta$
e)	$\varphi < -\rho < \rho < \theta < -\varphi$ $\varphi < \rho < -\rho < \theta < -\varphi$ $\rho < -\varphi < \varphi < \theta < -\rho$ $\rho < \varphi < -\varphi < \theta < -\rho$	f)	$\rho < \varphi < \theta < -\varphi < -\rho$ $\varphi < \rho < \theta < -\rho < -\varphi$
g)	$-\varphi < -\rho < \rho < \theta < \varphi$ $-\varphi < \rho < -\rho < \theta < \varphi$ $-\rho < -\varphi < \varphi < \theta < \rho$ $-\rho < \varphi < -\varphi < \theta < \rho$	h)	$-\varphi < \rho < \theta < -\rho < \varphi$ $\rho < -\varphi < \theta < \varphi < -\rho$ $-\rho < \varphi < \theta < -\varphi < \rho$ $\varphi < -\rho < \theta < \rho < -\varphi$
i)	$\rho < \theta < \varphi < -\varphi < -\rho$ $\rho < \theta < -\varphi < \varphi < -\rho$ $\varphi < \theta < -\rho < \rho < -\varphi$ $\varphi < \theta < -\rho < \rho < -\varphi$		

References

[1] H. Blum, *A transformation for extracting new descriptors of shape*. W. Whaten-Dunn (Ed.), Models for the Perception of Speech and Visual Forms. MIT Press, Cambridge, MA, 1967, 362–380.

[2] J. W. Bruce and P. J. Giblin, *Curves and Singularities*. Cambridge University Press, 1992.

[3] I. A. Bogaevsky, *Perestroikas of shocks and singularities of minimum functions*. Physica D. **173** (2002), 1–29.

[4] J. W. Bruce, P. J. Giblin and C. G. Gibson, *Symmetry Sets*. Proc. Roy. Soc. Edinburgh **101** (1985), 163–186.

[5] P. J. Giblin and S. Brassett, *Local symmetries of plane curves*. Amer. Math. Monthly **92** (1985), 698–707.

[6] P. J. Giblin and B .B. Kimia, *On the local form and transitions of symmetry sets, medial axes, and shocks*. Int. J. Computer Vision **54** (2003), 143–157.

[7] P. J. Giblin and J. P. Warder, *Reconstructions from Medial Representations*. Amer. Math. Monthly **118** (2011), 712–725.

[8] M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*. Graduate Texts in Mathematics, vol. 14. Springer-Verlag, F New York-Heidelberg, 1973.

[9] G. M. Reeve and F. Tari, *Mikowski symmetry sets of plane curves*. To appear in Proc. Edinburgh Math. Soc.

[10] K. Siddiqi and S. Pizer, *Medial Representations: Mathematics, Algorithms and Applications*. Springer: Computational Imaging and Vision, vol. 37, 2008.

[11] A. Saloom and F. Tari, *Curves in the Minkowski plane and their contact with pseudo-circles*. Geom. Dedicata **159** (2012), 109–124.